Metaverse (2023) Volume 4 Issue 1, *21 pages*.
doi: 10.54517/m.v4i1.2159

Asia Pacific Academy of Science Pte. Ltd.

REVIEW ARTICLE

A survey on geometric shape representation of objects based on medial axis transform

Junfeng Yao[1,2,3*], Hanyin Zhang[1], Xingxin Li[1], Mengyuan Ge[2], Weixing Xie[4], Yong Yang[1]

*[*1]Center for Digital Media Computing, School of Film, Xiamen University, Xiamen, China. E-mail: yao0010@xmu.edu.cn*

[2] School of Informatics, Xiamen University, Xiamen, China.

[3] Key Laboratory of Digital Protection and Intelligent Processing of Intangible Cultural Heritage of Fujian and Taiwan, Ministry of Culture and Tourism, Xiamen, China.

[4] National Institute for Data Science in Health and Medicine, Xiamen University, Xiamen, China.

ABSTRACT

Geometric shape representation algorithms are key technologies in the fields of computer graphics and geometric modeling. The Medial Axis Transform (MAT) is an important geometric model description tool that provides a simplified representation of complex geometric shapes while ensuring accurate descriptions of geometric shape and topology. Therefore, it can meet the requirements of many modern research fields, including geometric modeling, pattern recognition, model segmentation, model deformation, physical simulation, path planning, and more. This paper first introduces the basic concept of the medial axis transform, including the definition of the medial axis transform and the concept of medial axis primitives. It then describes the extraction algorithms for the medial axis transform, specific research on the medial axis transform in computer vision and computer graphics, potential applications of the medial axis transform, and medial axis transform datasets. Finally, the disadvantages and advantages of the medial axis transform are discussed, and some suggestions on possible future research directions are presented.

Keywords: medial axis transform; geometric shape representation; physical simulation

1. Introduction

Computer vision and computer graphics have always been hot research topics in the field of computer science. However, due to the increasing complexity of target scenes as technology advances, many works are constrained by the representation of geometric models, which hinders the overall computational efficiency. Therefore, efficient processing and representation of geometric objects are essential research topics. Some researchers thought of using contour tree[1,2] or skeleton[3,4] to simplify the representation of the original model in order to improve the overall computational efficiency. However, these representation methods still have some limitations, which can cause the loss of some feature information and affect the final effect. Similarly, the Medial Axis Transform (MAT)[5] is a very typical and popular geometric

ARTICLE INFO
Received: 20 April 2023 | Accepted: 5 May 2023 | Available online: 16 May 2023

CITATION
Yao J, Zhang H, Li X, et al. A survey on geometric shape representation of objects based on medial axis transform. Metaverse 2023; 4(1): 21 pages. doi: 10.54517/m.v4i1.2159

representation method that extracts the complete skeleton topology of 2D or 3D geometric shapes and records the thickness information of the corresponding parts of the original shape at each position on the skeleton. Finally, this simplified representation replaces the original shape to complete various tasks in computer vision and graphics. The purpose of this article is to introduce the relevant theory and applications of the medial axis transform and provide guidance to researchers in the computer vision and graphics communities, while also discussing some open issues, with the aim of encouraging more people to participate in research on the medial axis transform.

1.1. History and development of MAT

The concept of medial axis transformation was first introduced in 1976. At that time, Blum et al.[5] proposed the 2D medial axis transformation for extracting skeletal features of biological organisms, with research focusing on how to compute a unique and continuous medial axis from a 2D image. Later, Nackman et al.[6] extended the concept to 3D space, making the medial axis transformation applicable to more complex scenarios. Subsequently, research on the medial axis transformation mainly divided into two branches. One continued to investigate how to extract a medial axis with smaller errors, with Li et al.[7] proposing the concept of medial axis primitives, for example, and providing new ideas for medial axis transform extraction research. The other branch focused on various applications of the medial axis transform, utilizing its properties to great effect in computer graphics and computer vision. Sonka et al.[8] used the medial axis transformation for image semantic recognition, while Latombe[9] employed it in robot path planning. Moreover, the implicit local thickness information of the medial axis has been utilized in various fields, including shape segmentation[10,11], 2D or 3D curve/surface reconstruction[12,13], region decomposition in mesh generation[14,15], feature extraction in geometric design[16,17], shape approximation and retrieval[18–20], and so on.

1.2. Theoretical fundamentals of MAT

The medial axis transform is a mathematical tool used to describe 2D or 3D geometric shapes. Siddiqi et al.[18] provided a rigorous mathematical definition for shapes and their corresponding medial axis transforms. Simply put, the core of the medial axis transform lies in the inscribed sphere (circle). The medial axis is composed of a series of points inside the geometric shape, which must satisfy the following constraint: the inscribed sphere (circle) with these points as centers and the model boundary has at least two tangential points. These spheres are called medial axis spheres (circles), and the surface or curve formed by their centers is called the medial axis surface (line). The medial axis transform is a function composed of the medial axis and the corresponding inscribed sphere radius at each position.

The skeleton is a concept very similar to the medial axis. In the 2D case, the model's skeleton is identical to its medial axis, both of which are curves inside the model. In the 3D case, the medial axis may be composed of either curves or surfaces, whereas the skeleton can be entirely composed of curves, called the skeleton line. The following **Figure 1** shows an example of the medial axis in 2D.

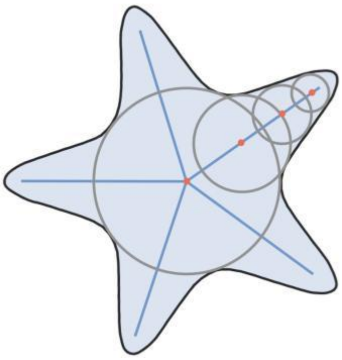

Figure 1. Medial axis transform of a 2D model.

Interpolation-based shape approximation is a common method in geometric approximation. In Zbrush[21], a sphere skeleton can be manually constructed to approximate the overall geometric contour, achieving a rough expression of the geometric shape. Building on this, B-Mesh[22] extracts a mesh from the sphere skeleton. Furthermore, the Sphere Mesh Model[23] uses interpolation based on endpoint spheres to approximate large-scale geometric volumes and achieve a good volume approximation effects. This approach has also been applied in progressive medial mesh simplification methods[24] and HEM[25] methods, which use a mesh composed of sphere interpolation to simplify medial axial expressions.

Based on this, the medial mesh[7] is another definition of medial axis transformation in three-dimensional geometry. It can be represented as a non-manifold triangular mesh called the medial mesh, which consists of triangles and isolated edges. Each vertex in the mesh represents a medial sphere and is represented by a four-dimensional coordinate like m = {c, r} where c ∈ R^3 is the center of the medial sphere and r ∈ R is the radius of it. Edge *e* = $\{m_i, m_j\}$ on the mesh will be used to represent the medial cone and the medial cone is obtained by linearly interpolating between two medial spheres at the two endpoints, and it can be expressed by the following formula:

$$\{m|m=\alpha m_i+(1-\alpha)m_j,\ \alpha\in[0,1]\} \tag{1}$$

Similarly, triangular faces on the mesh denoted by $f_{ijk}=\{m_i, m_j, m_k\}$, are referred to as medial slabs, which are obtained by interpolating the centroids of the three medial spheres at the vertices. The formula can be expressed as:

$$\{m|m=\beta_i m_i+\beta_j m_j+(1-\beta_i-\beta_j)m_k,\ \beta_i\in[0,1], \beta_j\in[0,1-\beta_i]\} \tag{2}$$

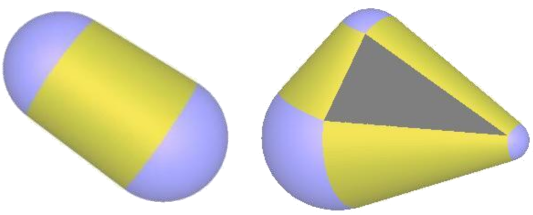

Figure 2. Illustration of medial cone and medial slab (**left**) medial cone (**right**) medial slab[61].

The medial axis transformation of a 3D shape is the approximation of the geometric model by the union of the envelopes of the medial primitives. **Figure 2** illustrates the medial cone and medial slab in the volume primitives.

In summary, as a skeleton of geometric models, the medial axis transform not only provides the topological properties of the model, but also describes the shortest distance from the skeleton to the boundary of the region. It expresses both shape and volume and has the following properties. (1) Uniqueness: the skeleton of the model is unique. (2) Symmetry: due to the symmetry of the medial spheres, the resulting skeleton also has symmetry. (3) Topological invariance: for a given model, its geometric topology remains unchanged and the corresponding medial axis is unique and invariant. (4) Dimension reduction: the medial axis of a 2D shape is a curve, while the medial axis of a 3D shape can be a curve or a surface. All of these properties match well with the requirements of some areas in computer vision and graphics.

1.3. Organizations

The remaining organization of this paper is as follows: Chapter 2 introduces the techniques for extracting the medial axis of geometric models, including precise computing methods and simplification methods. The latest research also supports the extraction of static or dynamic medial axes from point clouds or images. In Chapter 3, in combination with the characteristics of medial axis transformation, existing research on medial axis transformation techniques is presented. In the field of computer vision, it mainly includes semantic segmentation and mesh reconstruction, while in computer graphics, it mainly includes geometric deformation and collision detection. Chapter 4 introduces the future application areas of medial axis transformation and proposes some ideas. Chapter 5 introduces the existing datasets of medial axis transformation and hopes to build a more extensive dataset to expand the medial axis transformation database. Chapter 6 summarizes the advantages and disadvantages of medial axis transformation based on existing research and proposes future research directions accordingly. Finally, in Chapter 7, the article is concluded, and it is hoped that more scholars will contribute to the research on medial axis transformation.

2. Computing and approximating of MAT

Figure 3. Comparison of "spikes" in medial axis transform. (**top**) Medial axis transform with lots of spikes (**bottom**) Simplified medial axis transform.

As an ideal geometric approximation representation, the medial axis transformation can not only preserve the complete geometric shape and topological structure, but also approximate the original geometry's regional volume using local thickness information. However, in reality, the medial axis transform computed by using accurate medial axis transform computing algorithms often has defects and cannot meet expectations. This is because there is often some noise or small perturbations at the boundary of the geometric model, and in this case, the extracted medial axis transform will produce many unstable branches, which are also known as "spikes". These "spikes" usually appear in the form of separately hanging line segments or elongated triangles around the main body of the medial axis in geometry. The "spikes" of the medial axis transformation are shown in **Figure 3**.

Moreover, the existence of these "spikes" not only contributes nothing to shape representation but also takes up a large amount of storage space. Therefore, subsequent research has focused on the simplification of medial axis transformations which can also be called approximated MAT, overcoming the instability caused by the sensitivity of medial axis transformations to noise, and obtaining more stable medial axes. Generally speaking, we will use "medial axis transform computing" to refer to the extraction of accurate medial axis transform, and use "approximate medial axis transform" to refer to the extraction of simplified medial axis transform. This section mainly introduces some precise medial axis transform extraction methods and approximate medial axis transform simplification methods, which are mainly suitable for static model medial axis computing and require dense sampling of the geometric model surface. Finally, it is supplemented by the medial axis transform simplification algorithm for dynamic models and how to extract medial axis transform from special geometric models such as sparse point clouds.

2.1. Computing of MAT from mesh

For simple models such as planar polygons[26] or arc-boundary shapes[27,28], accurate medial axis transform can be conveniently extracted. The extraction algorithms of accurate medial axis transform mainly include topological thinning methods[29–33], distance field-based medial axis transform extraction algorithms[34–40], and medial axis transform extraction algorithms based on Voronoi diagram, etc. **Table 1** presents the similarities and differences as well as advantages and disadvantages among different methods for computing the medial axis transform.

Topological thinning methods[29–33] construct the medial axis transform using voxels as elements. Starting from the boundary of the original geometry model, the method diffuses inward until the local shape is refined and converged to a single voxel size, and then constructs the medial axis transform using these voxels. However, due to the resolution limit of voxels, the medial axis transform extracted by such methods is difficult to accurately represent the original geometric shape.

Table 1. Comprasion of different methods of MAT computing

Methods	Similarities	Differences	Advantages	Disadvantages
Topological thinning	Computing accurate MAT Slow computing	3D	Easy to convergence	difficult to represent original shape.
Distance field		2D & 3D	Reducing "spikes"	generate redundant medial axis spheres
Voronoi graph		2D & 3D	Good stability	produces redundant medial spheres
SMAT		Binary image	Globally smooth MAT	Only for simple images of objects

Distance field-based medial axis transform extraction algorithms[34–40] first construct a signed distance field function of the geometry model, and then deduce the positions and corresponding radii of the medial axis spheres from the distance field, thus obtaining the complete medial axis transform. In fact, the extracted medial axis transform is the medial axis of the iso-surface of the distance field function, and this method essentially reconstructs a new surface from the original model surface and then extracts the medial axis transform from the new surface. This approach is suitable not only for complex geometric shapes but also for significantly smoothing the noise in the original model and reducing the number of "spikes". However, constructing the distance field itself increases the computational cost, slows down the efficiency of medial axis transform extraction, and may generate too many redundant medial axis spheres due to the accuracy of surface reconstruction.

A Voronoi-based medial axis transform extraction method requires a smooth surface model and dense sampling on the model surface, which is saved as a point set. The Voronoi diagram is then constructed from the points to determine the conditions for constructing the medial axis transform. Lee et al.[26] proposed an algorithm for extracting the accurate medial axis transform of 2D shapes using Voronoi diagrams and edge bisectors, but it can only be used for 2D polygonal models. Attali et al.[41] applied Delaunay triangulation to sample points on the geometric surface and classified the resulting tetrahedra, retaining the tetrahedra that met certain requirements and removing those that did not affect the topology. The simplified set of retained tetrahedra is used to obtain the model's medial axis transformation. However, this method's effectiveness is affected by the number of sampled points, and efficiency decreases significantly with a large number of points. Bisssonnat[42] used Delaunay triangulation to sample points and selected the tetrahedra inside the model to generate the Voronoi diagram vertices, which are then used as medial axis points. The topology of the medial axis transform is then completed based on the connection relationship between the tetrahedra. Amenta et al.[43] proposed Power Crust, which removes non-real medial axis points from Voronoi vertices. The Power Crust algorithm not only supports various types of geometric models as inputs, such as triangle meshes or dense point clouds, but also has good stability. However, this method often produces redundant medial spheres, resulting in a redundant medial axis transform that cannot effectively solve the problem of "spikes".

Influenced by Zhu et al.[44] idea of spline fitting the medial axis transform, Wang et al.[45] proposed an algorithm called SMAT for accurately extracting the medial axis transform from binary images. SMAT is based on an iterative method consisting of two steps: segmentation and fitting. First, the image is segmented into several sub-regions, and the medial axis transform variation for each local range is obtained. Then, the local medial axis transforms are fitted with B-splines to obtain a globally smooth medial axis. SMAT can significantly reduce image size while maintaining a certain image quality, but for complex images of objects, it may lose many image details after compression.

2.2. Approximating of MAT

Table 2. comprasion of different methods of classical MAT approximating

Methods	Similarities	Differences	Advantages	Disadvantages
Angle-based method		Threshold is angle	effectively eliminate spikes	Changing the topology of MAT
λ-medial axis method		Threshold is medial sphere radius	Preverving topology	Cannot elimate spikes effictively
Scale Axis Transform	Approximate MAT Slow Approximating Reduce the number of medial spheres or medial edges	Threshold is a removal factor	Esay to reduce the medial sphere	Cannot control the number of remaining spheres
Prrogressive medial axis filteration		Based on edge merging	Preserving topology	Cannot update the position of medial sphere dynamically
Hausdorff error based method		Based on single-side Hausdorff distance	Controlling the approximating errors	Cannot update the position of medial sphere dynamically

Table 3. comprasion of different of methods of computing medial mesh

Method	Similarities	Differences	Advantages	Disadvantages
Q-MAT		Based on QEM to mearge edges	Fast computing and preserving topology	Manually defining ratio to merge the edges
Error thickness on medial axes		Considering the global external measurements	Easy to distinguish noise and features	Complex computing process
Q-MAT+		Increasing local sampling for external features	Better approximating effects	Dense sampling
LSMAT	Approximating MAT Slow Approximating Based on medial mesh	least-squares method	Obtaining the normal vectors of MAT	Dense sampling
Restricted Power Diagram		Considering the internal features	Achieving high-precision approximation	Inefficiency of construction of restricted power diagrams
Coverage Axis		Consider local approximation error and the global shape structure	Achieving high-precision approximation of MAT	Complex computing process

Based on the methods above, it is unrealistic to extract the exact medial axis transform of complex geometric models. No matter how the algorithm is processed, it cannot effectively solve the problem of “spikes” caused by medial axis transform redundancy. Therefore, the focus of many algorithms is to extract a approximated medial axis transform, which reduces the number of medial spheres or edges while ensuring that the medial axis transform closely approximates the original model. **Table 2** presents the similarities and differences as well as advantages and disadvantages among different classical methods for approximating the medial axis transform, which is based on the definition of the mathematical principles. And **Table 3** presents the other one for approximating medial axis transform based on the definition of medial mesh.

There are two kinds of approaches to angle-Based methods[46–49]. The first kinds of method determines the angle between each medial axis point and its two nearest geometric surfaces. When the angle is smaller than a threshold θ, the corresponding medial sphere is removed. The other kind of method is to sort each medial sphere by the corresponding angle and sequentially eliminate a specified number of medial spheres from smallest to largest. This method can effectively eliminate “spikes”, but may cause changes in medial axis transform topology, and threshold adjustment is also important. If it is too large, the medial axis transform cannot approximate the geometric model, and if it is too small, “spikes” cannot be effectively removed.

Similarly, the external sphere radius-based medial axis transform simplification method[46–50] (λ-medial axis method) constructs an external sphere with the closest boundary point for each medial axis point and removes all medial spheres with a corresponding external sphere radius smaller than a threshold λ. Chaussard et al. [51,52] proved that when a small threshold is selected, the complete topology structure can be preserved. However, a threshold that is too small still cannot effectively eliminate “spikes”, and a threshold that is too large will cause details of the medial axis transform to be lost.

The Scale Axis Transformation [53,54] (SAT) sets a removal factor “S” and enlarges the radius of all medial spheres by “S” times. The medial spheres that are completely surrounded by other medial spheres after being enlarged are removed, and the remaining medial spheres are reduced to the current “1/S” radius to obtain a simplified medial axis transform. Similarly, setting a removal factor that is too large will damage the medial axis transform topology and lose local details, and it is still impossible to control the number of remaining medial spheres.

The Progressive Medial Axis Filteration[24] (PMAT) improved the Scale Axis Transform algorithm by adopting an edge merging approach. The algorithm calculates the ratio of the length difference between each medial edge and the radius of its two medial spheres, and merges the medial edges in ascending order to obtain a simplified medial axis transformation. However, the algorithm does not dynamically update the radius of the current medial spheres during the merging process, only referring to the overall state before merging. This method therefore cannot solve many of the problems in the Scale Axis Transformation.

In the Hausdorff Error-based method[25] (HEM), the idea of edge merging is also used. The medial edges are merged, and the single-sided Hausdorff distance between the original model and the simplified medial axis after edge merging is computed as the cost for each edge merging. This process continues until the preset error range is approached, resulting in a approximated medial axis transformation. However, this method is inefficient because it requires a large number of Hausdorff distance calculations, and it cannot guarantee obtaining the optimal medial axis sphere because the simplification reference standard is based on the Hausdorff distance. Zhu et al.[44] and others used the Hausdorff distance as a threshold for pruning the extracted medial axis transform to improve the stability of the medial axis transformation. Then, the stable medial axis transformation is approximated by a spline curve to obtain a smooth and compact medial axis transformation representation. This method can be used to approximate the medial axis transform of any geometric object, but it may require repeatedly adjusting the threshold to obtain the desired medial axis transformation, and it requires the geometric boundaries to be sufficiently smooth and free of noise. Sun et al.[55] extended this algorithm to 3D models, and in the process of extracting the approximated medial axis transformation, not only the edge merging cost described by the one-sided Hausdorff distance is considered, but also the topological structure of the medial axis transformation. The approximated medial axis transformation is extracted through several iterations. However, this method still cannot handle models containing noise or missing point cloud models well.

The Q-MAT method[7] borrows the idea of Progressive Medial Axis Filteration and represents the medial axis transform as a medial mesh. The algorithm is based on the most classic Quadratic Error Metric (QEM) in mesh simplification and redefines the quadratic metric error based on the medial mesh. By setting the minimum value of this error, a new medial sphere is obtained after merging the medial edges. Q-MAT introduces a stability ratio as the cost of edge merging and merges medial edges in order from small to large, ensuring that “spikes” are removed first.

However, all of the aforementioned methods only consider the least important medial spheres or medial edges based on local information of the medial axis transform, ignoring the impact of these local features on the global shape, thus disregarding some important characteristics of the models. Therefore, Yan et al.[56] proposed a simplification method based on global measurements, which requires dense sampling and sets an Erosion Thickness(ET) to identify important features or noise, resulting in a more complete and accurately approximated medial axis transform. Q-MAT+[57] increases sampling on local slabs of the surface to obtain a more precise medial axis transformation. LSMAT[58], the least-squares medial axis transformation algorithm, also extracts the static medial axis transform from dense sampling. The algorithm calculates curvature using the normal vector of each sampling point, constructs quadratic surfaces using least-squares method, and takes the intersection points of adjacent surfaces as medial points. The normal vectors of medial points are interpolated based on the normal vectors of the quadratic surfaces at the medial point. The algorithm obtains the normal vectors of the medial axis transform but increases computational burden and is not suitable for geometric models with complex shapes.

From another perspective, the aforementioned methods only consider the external features of the medial axis transformation, including sharp surface features and corner information of the input mesh, but ignore internal features such as connections and topology between the medial axis transforms. Sampling for internal features during medial axis transform approximation is insufficient, and thus, the correct topological connectivity between medial axis transforms cannot be well represented. Wang[59] et al. innovatively introduced medial spheres with restricted power diagrams[60]. The dual structure of these restricted power diagrams represents the connectivity between medial spheres and the tangent surface areas of these medial spheres. The algorithm first calculates the tangent points of the medial spheres to construct restricted power diagrams, classifies the medial spheres based on this information, identifies the features of the medial spheres, and places them in the correct position, such as at the intersection point. This algorithm is an efficient method for sampling and preserving features of the medial axis transform internally, achieving high-precision approximation of the medial axis transformation. However, the construction efficiency of the restricted power diagrams affects the extraction time of the medial axis transform. Additionally, this algorithm cannot be used for medial axis transform approximation of 2D models and is only applicable to 3D models.

Dou et al.[61] proposed Coverage Axis for 3D shape skeletonization, which considers not only local approximation error but also the global shape structure. It can effectively alleviate the negative impact of noise and approximate a compact and expressive Medial Axis Transform. This method not only supports triangle meshes as input but also extracts the medial axis transform of other geometric structures, such as point clouds and poor-quality meshes. However, because the core idea of this method is to find as few inside medial balls as possible, the calculation process is complex, and the computational efficiency is still relatively low.

2.3. Extracting dynamic MAT from triangle mesh

As an important research topic in computer graphics, animation approximation aims to approximate the original animation sequence with a simplified structure as closely as possible. In this problem, it is necessary to approximate a dynamically changing geometric model with medial axis transform.

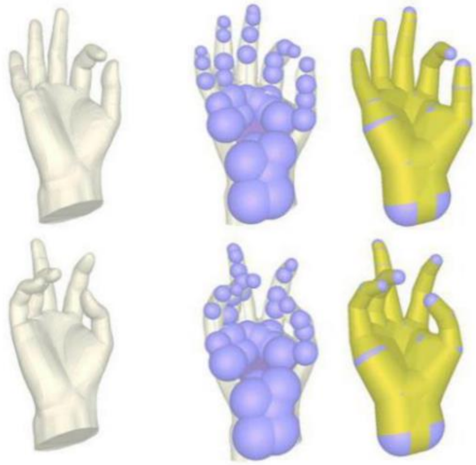

Figure 4. Extract dynamic medial axis transform from triangle mesh[62].

Pan et al.[57]. proposed a deformable medial axis transform for dynamic surfaces. **Figure 4** shows the extraction of dynamic medial axis transform from a triangular mesh. First, the precise medial axis transform of the reference frame is calculated and used as a reference to further calculate the correspondence between the sample point sets of each deformation sequence and the volume elements of medial axis transform. Then, the As Rigid As Possible (ARAP) transformation of the medial axis transform during the animation process is constructed to obtain the medial axis approximation of the dynamic mesh. The algorithm dynamically adjusts the number and size of medial spheres while ensuring the topological structure of the medial mesh, making it possible to approximate dynamic sequences well. However, this method cannot be applied to animation sequences with mesh topology changes, nor can it extract dynamic medial axis transforms from collapsed mesh sequences.

2.4. Extracting MAT from sparse point clouds

Previous medial axis transform computing or approximating algorithms only targeted geometric bodies with dense sampling, and were unable to accurately extract the medial axis transform of geometries represented by sparse sampling or sparse point clouds. However, sparse point clouds are a very common way of representing geometry in the real world. Inspired by MAT-Net[63] and P2P-Net[64], Yang et al.[65] proposed P2MAT-Net, which uses a deep neural network to learn the transformation from a point cloud to an approximate medial sphere. They use the neural network to map sparse point clouds into medial spheres, and adjust the predicted medial spheres using a sphere boundary strategy and a normal optimization strategy. They also learn the connection structure between medial points from the point cloud to obtain a complete medial mesh. The algorithm achieved good results in noisy point clouds with very few points, effectively learning point cloud features and extracting stable medial axis transform. However, during the process of predicting the medial sphere, the algorithm moves the point cloud position one by one to obtain the medial sphere. **Figure 5** shows the effect of extracting medial axis transformations from sparse point clouds. For tasks where the number of medial spheres is less than the number of point cloud models, the algorithm cannot be directly applied to solve this problem. Iterative medial axis transform extraction algorithm[66] is also an algorithm for extracting medial axis transform from sparse point clouds. IMAT[66] solves two constrained sub-problems in each iteration, including shape marching, finding the best position to place the skeleton sphere on the geometric boundary (extracted from the point cloud), and determining the radius of the medial sphere. Specifically, the error function between the sphere and the geometric boundary is minimized, and the number of spheres is continuously adjusted until the error function converges. The medial axis transform extracted by this algorithm is relatively robust, but the convergence speed is slow, which affects computational efficiency.

Figure 5. Extract static medial axis transform from sparse point clouds[65].

3. Research on technology based on MAT

Computer vision and computer graphics are two very popular fields in computer science research nowadays. Both of them are inseparable from the description of geometric models. The ability to find an accurate and concise geometric representation is an effective way to improve efficiency and effectiveness. The medial axis transform of geometry can be well integrated into current visual and graphical applications and is a good solution to some hard-to-break-through dilemmas.

This chapter is divided into two main subchapters, including two sections on computer vision and computer graphics. Each is further divided to show specifically how the medial axis transform can break through specific dilemmas in the field.

3.1. MAT technology in computer vision

Mesh reconstruction

Mesh reconstruction is the most direct and essential application of medial axis transformation. It can be said that almost all extraction of medial axis transform is in order to better reconstruct complex geometric shapes. In addition, in recent years, deep learning can be used well in 2D image-related research scenarios, but direct application of this technology in 3D model representation is difficult. This is because, for 3D model surface sampling points, the spatial and topological relationships between them are complex and unpredictable. Traditional 3D reconstruction algorithms are multi-view representation[67], voxel reconstruction[68], or deep learning based on point clouds[69]. Multi-view reconstruction represents 3D geometry as 2D images from multiple angles, but this does not identify problems such as self-obscuration implicit in the input image. Conversely, a voxel is a spatial grid, which is a method of uniformly dividing space. These small voxels can work well as input to a neural network, but the computational overhead increases as the more spatial resolution is required. Point clouds are also suitable for deep neural networks for learning, but this representation does not capture the topological features of the 3D model well, resulting in poor reconstruction quality. Therefore, a special convolutional neural network is needed that can learn local features based on the topology to accomplish shape perception while keeping the computational cost low. In fact, the medial axis transform can meet the above requirements very well. As mentioned before, the medial axis transform is the geometric model's exact skeleton and contains the local radius to represent the thickness information. These properties can well represent the geometric topology and accurately restore the original shape of the target. At the same time, the medial axis transform is a representation that is consistent with human perception of shape[70,71]. Based on this, Hu et al. proposed MAT-NET[63]. This is the first algorithm to learn the medial axis transform features using deep neural networks to achieve 3D reconstruction. Hu et al. first extracted their corresponding medial axis transform as a dataset based on the 3D models provided in ModelNet40[68] for classification tasks. After that, the network is constructed by referring to the ideas of PointNet[69] and PointNet++[72]. Specifically, the new CNN is divided into two parts, Group-MAT and Edge-net. First, Q-MAT[7] is used to obtain the medial axis mesh, after which PointNet is used to learn the features of discrete medial axis spheres and encode them into the Group-MAT network. Group-MAT calculates the local data structures of these disordered medial axis spheres based on the connection information of edges and then passes them into Edge-net to obtain the local features of these structures, and finally completes the learning of surface features to realize the reconstruction from medial axis transform to geometric model. This method can reconstruct the accurate target model well, but the computation time is still long and requires a large amount of input data for support, which does not perform well for complex scenes.

In recent years, in addition to the reconstruction of surfaces from multi-view images mentioned above, there has been some research on how to reconstruct 3D models from single-view images. Relying on the rich dataset of 3D models nowadays[73], existing deep networks can reconstruct various types of geometric models from single-view images[74–81]. However, none of these representations can perceive the topological details of the geometric structure well, and can only restore the original geometric information. Therefore they can only reconstruct geometric models with simple topology or shape. For neural networks to fully learn the topological information of geometric shapes, skeleton-based methods[82,83] were proposed, which can accurately capture the topology of the original model, but their generated skeleton points cannot provide the complete surface geometric information, they cannot directly construct surfaces. And the medial axis transform can solve this problem well. First, the radius information provided by the medial sphere of the medial axis transform can represent the local thickness of the geometry, which is very important geometric information in the reconstruction; second, the connectivity of the medial sphere can also represent the topology of the complex geometry well, with two spheres forming an edge between them and three spheres connecting into a surface; finally, the existing medial axis transform can be directly recovered into a flowing shape by the Marching Cubes[84] algorithm triangular mesh. Based on this idea, Hu et al. proposed the IMMAT network[85] to predict the medial sphere and skeleton of the geometry from the single-view input image and then further reconstruct the surface. The whole network consists of two main modules: the image2sphere module and the topology prediction module. The former, as the name means, is used to learn features from the input single-view image and predict a set of discrete median points and corresponding radius. The latter predicts the topological connectivity relationships between these spheres. Finally, these spheres are smoothed using the MAT Smoothing module, and the final surface is reconstructed from the medial axis transform. This method is the first algorithm that uses supervised learning to generate the medial axis transform and surface reconstruction from single-view images; however, due to the limitations of single-view images, such methods cannot handle some complex geometries, while such algorithms are inherently sensitive to noise, and excessive noise can affect the reconstruction results.

Similarly, the medial axis transformation reconstruction technique can be used in fields such as metallurgy. The casting process allows for obtaining components with approximate shapes[86–88]. This process requires the estimation of the mold filling and melt solidification times to design the casting system and, if necessary, to make adjustments to the component geometry, where the evaluation of the solidification process can be done either by FEM-based methods[89] or by geometric methods. Among them, the FEM (Finite Element Method) analysis requires solving thermodynamic equations and a complex computational process, while in the geometric approach, the estimation of the time is achieved by evaluating the geometry when geometric factors are the main ones affecting the solidification time of the casting. Therefore, some methods based on the medial axis transform were proposed for geometric analysis[90–92]; however, these methods can only specify the metal solidification path manually and cannot be automated, and at the same time, they can only be used to estimate the geometry but not to optimize it. Therefore, a new topology-optimized casting algorithm based on the medial axis transform is proposed[93]. The algorithm first generates a geometric model based on the optimized topology. After that, the simplified medial axis transform of the model is calculated, and the local thickness information of the medial axis transform is used to mark the local radius maximum and define the curing direction automatically. Finally, the maximum radius is gradually adjusted and the surface reconstruction of the casting is completed.

Shape segmentation

Shape segmentation is a technique that automatically decomposes a complex geometry into several independent and simple shaped components using some algorithm and can be used in areas such as modeling[94,95], and texture mapping[96]. The majority of current shape segmentation algorithms fall into two main categories. One of them is semantic segmentation methods based on supervised learning and labeled datasets, which split the geometry into a number of predefined labels, but do not provide specific shape information for each label, which may lead to segmentation errors. The other category is geometric analysis methods which accomplish the segmentation task based on rules or unsupervised learning. Geometric analysis algorithms need to find the boundaries and ensure that the geometry within the boundaries matches the features[97]. Existing 3D shape segmentation algorithms based on geometric analysis cannot extract features of complex geometric components well[98–100], and at the same time, the extracted local geometric features lack global contextual information, which may lead to unexpected counterintuitive results. Therefore, Lin et al. proposed Seg-Mat[101] to accomplish geometric segmentation using the medial axis transform. The algorithm first extracts the medial axis mesh using the Q-MAT[7] algorithm to avoid boundary perturbations that generate spike to affect the final result. After that, the medial axis mesh is structured and a rough initial segmentation is performed to segment the parts with dimensional changes and non-fluid shapes, as well as the parts with sharp radius changes based on local thickness information. After that, the spatial region is processed based on the region growth algorithm[102] to complete the fine segmentation. Finally, the segmentation result of the medial transformation is reduced to a triangular mesh[100,102–104]. The current algorithm only supports simple geometric object segmentation, which cannot be applied to complex scenes, and also requires high quality of the extracted medial axis to minimize spike interference and also to control the threshold value to achieve more accurate segmentation results.

3.2. MAT technology in computer graphics

Physical simulation

The motion and deformation of each animated object in the physical simulation are computed independently, so some areas of the animated object may be updated to the same spatial location. This phenomenon is known as collision, which is visually manifested by different geometric elements (e.g. triangles) overlapping or penetrating each other. Obviously, collisions are a serious visual flaw, and in severe cases, it may appear that one animated object enters or passes inside another animated object. Collisions can occur not only between different animated objects in the scene but also between different local areas of the same animated object. The former is called other collision between objects, while the latter is called self-collision of objects. The computational task of collision processing mainly includes two aspects: collision detection and collision response. Collision detection not only determines whether a collision has occurred but also returns a list of all geometric elements associated with the collision and gives collision information such as the puncture depth. The collision response is based on the returned collision information and corrects the geometry elements with wrong motion states in a reasonable and efficient way.

Lan et al.[105] proposed a medial-elastic algorithm for discrete collision detection. medial-elastic driven discrete collision detection performs intersection tests on medial-elastic elements as enclosing elements and then queries the pre-defined 3D space by computing the voxel index at the deepest point of intersection[106] to collect all potential collision triangular elements. Finally, a small number of triangle intersection tests are executed to obtain the location of specific collision occurrence. This method takes advantage of the ability of the medial mesh to express the 3D model shape as well as the internal skeletal features and maps the high-resolution 3D mesh onto the medial mesh by bisecting and weighting it to simplify the global solution step in the dynamic projection method. The key idea is similar to the fully simplified dynamic projection method proposed by Brandt et al.[107]. It demonstrates that the medial axis elements can be used as enclosing elements in discrete collision detection to improve detection efficiency. In terms of determining collision culling, it innovatively proposes a strategy to test the surface distance function from boundary conditions to determine whether the medial axis elements intersect. However, the discrete collision detection based on the medial axis also suffers from the common problem of other methods, it is difficult to accurately capture the collision between objects or geometric elements moving at high speed. It is also mentioned in its experimental results that the time step is not small enough to miss the collision and lead to the penetration phenomenon between models. Also, the collision culling during the simulation would be a violent traversal of the medial elements of all the simulated objects without a higher-level collision culling method, so there would still be performance issues with a high number of simulated objects.

Figure 6. Elastic body collision based on Medial-IPC[107].

On the basis of the discrete collision detection idea based on Medial axis transform, Lan et al.[108] further proposed a continuous collision detection framework Medial-IPC based on medial mesh. Among them, the continuous collision detection module is implemented based on the IPC[108] algorithm, which can ensure that the penetration phenomenon will not occur at all times during the movement of the object. Specifically, the IPC algorithm is based on the theory of incremental potential energy. Under the condition of ensuring that the kinetic energy, elastic potential energy, and frictional potential energy of the system all decrease as a whole, the algorithm solves the energy equation with “non-penetration” as a constraint. In order to solve this nonlinear system, the constraint of “non-penetration” was innovatively defined as a barrier energy and represented by a logarithmic function. This ensures that when two objects are close enough, a sufficiently large contact potential energy will separate them, and when the objects are far enough apart, the contact energy is very small or even zero. This is why objects will never penetrate each other. In Medial-IPC, the same idea as medial-Elastic is used to replace collision culling algorithms such as Spatial-Hash by Medial axis mesh to improve computational efficiency. At the same time, according to the characteristics of IPC, the distance calculation method between medial axis primitives and the energy of barrier function are defined to ensure the principle of non-penetration. The algorithm makes good use of the characteristics of the medial axis transform to accelerate the IPC calculation process, and improves the calculation efficiency of the continuous collision detection of elastic bodies. However, due to the computational complexity of the medial axis transform, the algorithm cannot be well applied to real-time simulation applications. **Figure 6** shows the collision effect of the elastomer based on the Medial-IPC framework.

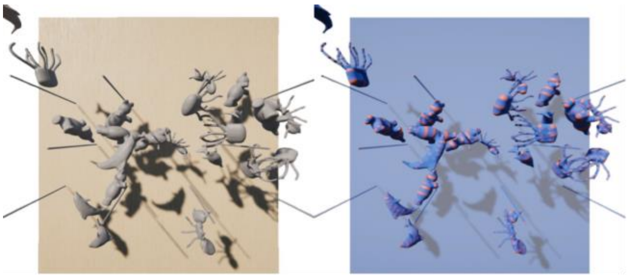

Figure 7. Continuous collision detection based on medial axis transform in rigid body simulation[109].

Song et al.[110] proposed a continuous collision detection method driven by the medial axis. This method makes full use of the geometric characteristics of the medial axis primitives, simplifies the complex continuous collision detection problem into two sub-problems that can be solved analytically, and obtains the first collision time by fast convergence through alternating iteration. Under the premise of fully ensuring the accuracy, the computational efficiency of continuous collision detection in rigid body simulation is improved. The complex root finding problem of high-order polynomial is simplified into the problem of nearest sphere pairs between medial axis elements and the problem of continuous collision detection between medial axis spheres by means of fixed parameters, and a new alternating iterative numerical solution method is proposed. Then, the problem solution is extended to the collision detection between different types of medial axis primitives, and an optimization scheme is proposed to avoid redundant primitive testing for complex primitives. **Figure 7** illustrates the rigid body collision simulation based on medial axis transform.

Geometric deformation

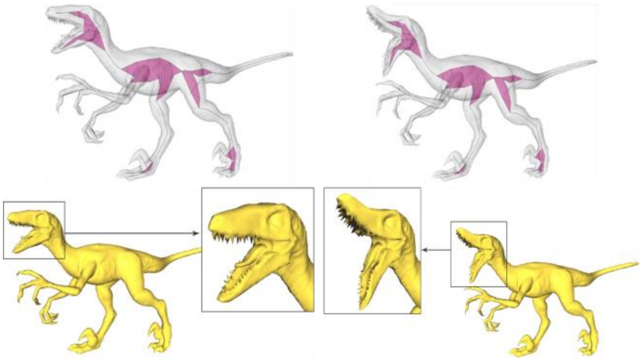

Figure 8. Deformation and volume preservation of ARAP driven by the medial axis[111].

Because the medial axis computation is sensitive to noise, early work made it difficult to obtain high-quality medial axes that were simple in structure (no undesirable spikes), approximated the surface precisely, and was compact enough to compute deformations. Fortunately, with recent advances in neutral axis simplification, such as Q-MAT[11], high-quality medial axes can be obtained by pruning unstable branches and simplifying from an initially poor-quality medial axis. As a result, it is now possible to use a "real" medial axis to drive shape deformation. Lan et al.[110] proposed an algorithm for shape deformation driven by the medial axis, a deformation scheme "As Rigid As Possible" (ARAP) is used to deform the medial axis so that its local transformation is as close to the rigid transformation as possible. Surface features of the deformed shape are obtained using an implicit skinning-based approach. The algorithm is essentially a skeleton structure for representing 3D shapes, but it provides more information about the surface, such as thickness and features, than a traditional bar skeleton or curved skeleton. Combining ARAP deformation with radius adjustment on the dial mesh ensures that the overall volume is maintained during shape deformation. Equal surface projections with tangential relaxation not only preserve surface features but also address the candy-wrapper and volume-loss artifacts in distortion and bending associated with traditional deformation methods. **Figure 8** shows the transformation effect of ARAP deformation and volume preservation driven by the medial axis.

4. Database based on MAT

In order to advance the research and applications related to the medial axis transform more widely, a complete dataset is essential. Especially in deep neural network-based applications[63,85,101], a canonical dataset is necessary to train a practical neural network.

Currently, only Hu et al.[63] constructed the first medial axis transform dataset ModelNet40-MAT based on ModelNet40[68]. No other related work on medial axis transform data is available. Therefore, more publicly available datasets of 2D or 3D geometric models, which can be based on point clouds or on triangular surfaces, can be searched for and the corresponding datasets constructed by extracting the medial axis transform of these shapes. Further, these datasets can be organized to construct a complete medial axis transform database to provide data support for future medial axis transform related research.

5. Open challenges and future directions

This chapter discusses the advantages and disadvantages of medial axis transform, and tries to lead to possible future research directions according to these characteristics.

5.1. Advantages and disadvantages of MAT

After the above description of the calculation of medial axis transform and the related applications of medial axis transform, the advantages and disadvantages of medial axis transform can be summarized.

The most prominent advantage of the medial axis transform is its simplicity; it can describe the complete geometric features of the original model with a simple data structure. Medial axis transform can preserve the complete topological characteristics of the original model with precisely defined skeletons. At the same time, the shortest distance from the local boundary provided by the radius can be used to restore the original shape. Furthermore, the medial axis mesh composed of medial axis sphere, medial axis cone and medial axis slab can provide better wrapability to the original model. Given these properties and the existing applications of the medial axis transform, it can be found that the most useful applications of the medial axis transform are deformation calculation, collision detection and shape analysis.

From a deformation calculation perspective, the medial axis transform itself acts as a skeleton located at the center of the model, a more general and precise definition of the skeleton than the common stick fire curve skeleton. As a simplified model for deformation calculation, the medial axis transform can maximize the ability to capture the local shape detail features of the model, especially for objects with complex geometric structures.

From the collision processing point of view, the volume envelope of the medial axis transform can highly reconstruct the model shape, while the volume envelope has a smaller amount of data and a more compact envelope effect compared to the hierarchical envelope box. This means that the medial axis transform can provide more efficient collision culling as a simplified model for collision processing.

From a shape analysis point of view, the medial axis transform cull the non-essential aspects of the shape and retains only the geometric elements that best represent its topology. Compared to other shape representation tools, the medial axis transform represents complex shape features, such as circles or sharp protrusions, with only a small amount of geometric data. More importantly, the medial axis transform contains a local volume description of the model, which directly characterizes the semantic information of the local thickness of the structure.

However, from the existing extraction methods of medial axis transform and related applications, it can be seen that the medial axis transform still has certain limitations, which makes it unable to be widely used. The first drawback is the instability of the medial axis transform. Because the medial axis transform is very sensitive to noise and subtle perturbations at the boundary of the model shape, these subtle perturbations and noises will cause many unstable branches called “spike” in the calculated medial axis. The computation of these “spike” is not only time-consuming, but also does not contribute to the expression of the final geometry and occupies a large amount of storage space. The existing extraction algorithms all try to ensure the accuracy of the medial axis itself as much as possible while extracting the medial axis transform with fewer spike. The second drawback is that the input model has strict requirements, and the current algorithm can only extract densely sampled point clouds or closed popular mesh. Although Yang et al.[65] proposed a medial axis extraction algorithm based on sparse point cloud model in 2020, the effect still needs to be improved. Finally, the medial axis transform is generally computationally inefficient, and it is very time consuming to compute the complete medial axis. The existing medial axis transform extraction technology has almost no algorithm to greatly optimize the computational efficiency, so that it can be calculated in real time, so that the medial axis cannot be well applied to real-time tasks.

5.2. Future research directions

For the extraction of medial axis transform, we can start from four aspects. On the one hand, the existing simplification algorithm of medial axis transform is improved to reduce the redundancy and instability of medial axis transform. The reserch topic can be “Approximating MAT by Reduce Spikes Efficiently”. On the other hand, we can start to improve the calculation speed of the medial axis transform, so that it can reach the standard of real-time application. Some topic basd this idea can be “A Real-time Method to Extract MAT”. In addition, we can also consider extracting medial axis that do not require an input model, such as sparse point clouds or medial axis transform of images. Similar topic can be “The Extraction Methods of MAT for Different Types of Input Models”. Finally, after the medial axis extraction algorithms of different models are available, we can consider constructing a corresponding medial axis transform dataset for the public geometric model dataset and organize it as a medial axis transform database. We can use something like “Building an Open and Comprehensive MAT Database ” to describe this topic.

For the application of the medial axis transform, the envelope of the medial axis primitives can be considered instead of the traditional bounding box. Therefore, any technique that originally required the application of bounding boxes can be further studied, such as “A Ray Tracing Acceleration Method Based on MAT”. At the same time, the idea of using the medial axis transform skeleton to represent the shape, as well as the local thickness information, can well meet the requirements of the existing 3D reconstruction tasks. In addition, for the simulation field of intelligent manufacturing, the medial axis transform can also be considered as a model expression to optimize the simulation process.

6. Conclusion

Medial axis transform, as a kind of precisely defined skeleton, can accurately express the topological structure of geometric shapes. The radius corresponding to the medial axis sphere represents the local thickness information of the geometric object, and the original geometric shape can be well restored from the medial axis transformation. The medial axis primitives defined by the medial axis mesh can provide a more compact envelope for the original model. These features can be well integrated into the mainstream research of computer vision and computer graphics. However, the computation of the medial axis transform is not only very time-consuming, but also may extract the medial axis transform with a large number of “spike” since the computation process is susceptible to noise or perturbation at the boundary of the initial model. These spike not only occupy a lot of storage space, but also make no contribution to the shape representation, increasing the computational burden. In addition, most of the existing medial axis transform extraction techniques only support smooth surfaces and densely sampled geometry, and there are few related studies on the calculation of medial axis transform under point clouds or other types of geometric representations. These shortcomings prevent the wide application of the medial axis transform.

Future research related to medial axis transformation can be divided into two main branches: medial axis transform extraction and medial axis transform application. For the research of medial axis transform extraction, we can focus on how to improve the calculation speed of medial axis transform. At present, there are few researches in this direction, which is an important factor that hinders the integration of medial axis transform into real-time applications. In addition, one can also consider extracting the medial axis transform of multiple types of geometry, and reasonably optimize the "spike". For the application of medial axis transform, the original geometric model can be replaced by medial axis transform to improve the efficiency or effect of related research, and the envelopedness of medial axis mesh can also be considered to replace the hierarchical bounding box related algorithm. Finally, for the emerging research direction of digital twin, there are many possibilities for the application of medial axis transformation. At present, there is no related research work combined with medial axis transformation in this field. This review can provide new technical ideas and theoretical support for applications such as industrial design, robot control, virtual reality, and human organ policy, and will certainly help to promote the research process of digital twin technology in China.

Conflict of interest

The authors declare no conflict of interest.

Acknowledgements

The paper is supported by the Natural Science Foundation of China (No. 62072388), Collaborative Project fund of Fuzhou-Xiamen-Quanzhou Innovation Zone (No. 3502ZCQXT202001), the industry guidance project foundation of science technology bureau of Fujian province in 2020 (No. 2020H0047), the natural science foundation of science technology bureau of Fujian province in 2019 (No. 2019J01601), the creation fund project of science technology bureau of Fujian province in 2019 (No. 2019C0021), and Fujian Sunshine Charity Foundation.

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