Mathematics and Systems Science 2025, 3(2), 3577.
https://doi.org/10.54517/mss3577

Asia Pacific Academy of Science Pte. Ltd.

Article

Solutions of higher-order fractional boundary value problems with fractional left-focal like conditions

Jeffrey W. Lyons

Department of Mathematical Sciences, The Citadel, Charleston, SC 29407, USA; jlyons3@citadel.edu

CITATION

Lyons JW. Solutions of higher-order fractional boundary value problems with fractional left-focal like conditions. Mathematics and Systems Science. 2025; 3(2): 3577. https://doi.org/10.54517/mss3577

ARTICLE INFO

Received: 9 April 2025
Accepted: 6 May 2025
Available online: 23 June 2025

Abstract: We consider a higher-order Riemann-Liouville fractional boundary value problem with two-point boundary conditions. The higher-order fractional conditions are left-focal inspired. Using fixed point results, the existence and nonexistence of positive solutions are conditioned upon the size of the parameter λ in the differential equation. Our approach involves constructing a Green function by combining the Green's functions of a fractional problem of lower order and a left focal boundary value problem. We then use induction to increase the order. An example is provided to illustrate the existence and nonexistence regions.

Keywords: Riemann-Liouville; left-focal; fixed point; existence; nonexistence; convolution; induction

2020 MSC: 26A33; 34A08

1. Introduction

Let $m, n \in \mathbb{N}$, $m \geq 3$, with $\alpha \in (m-1, m]$ and $\beta \in [1, m-1]$. Consider the following Riemann-Liouville fractional boundary value problem

$$D_{0^+}^{\alpha+2n} u(t) + (-1)^n \lambda g(t) f(u) = 0, \quad 0 < t < 1 \tag{1}$$

subject to the left-focal inspired fractional boundary conditions

$$\begin{aligned} u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, m-2, \quad D_{0^+}^{\beta} u(1) = 0, \\ D_{0^+}^{\alpha+2l+1} u(0) = D_{0^+}^{\alpha+2l} u(1) = 0, \quad l = 0, 1, \ldots, n-1 \end{aligned} \tag{2}$$

We stipulate $\lambda > 0$ is a positive parameter and $f : [0, \infty) \to [0, \infty)$ and $g : [0, 1] \to [0, \infty)$ are continuous functions such that $g(t) \not\equiv 0$ on $[0, 1]$. We are concerned with the existence and nonexistence of positive solutions to Equations (1) and (2).

Of interesting note is the alternating component of the nonlinearity. This is due to the sign-changing of the Green's function whenever the order is increased by a factor of 2. This added piece ensures that the Green's function remains positive for any choice of n. Additionally, we demonstrate through induction how to achieve any chosen higher order $2n$.

To that end, we construct the Green's function associated with Equations (1) and (2) following the procedure outlined in [1]. The idea is to convolve the Green's function $G_0(t, s)$ for a lower-order problem with the Green's function of a left-focal boundary value problem. Induction is subsequently implemented to increase the order of the Green's function. Next, we state properties of the lower-order Green's functions found in [2], and show that these properties are inherited by the higher-order Green's function.

Finally, we apply this framework in an implementation of the Krasnosel'skii Fixed Point Theorem.

The major impetus for this result comes from two works, namely [2] and [3]. In the former, Lyons and Neugebauer implemented the convolution of the Green's function for a fractional order boundary value problem with that of the Green's function for an ordinary boundary value problem. Subsequently, Neugebauer and Wingo increased the order of the fractional boundary value problem by factors of $2n$ by applying an induction argument. These two works in themselves are generalizations of the existence and nonexistence results from the early 2000s by Graef and all [4–6]. Their motivation initially was proving the existence of positive solutions to beam equations, which are fourth-order ordinary boundary value problems, using the Krasnosel'skii Fixed Point Theorem. However, in [7], the authors later generalized to an n-th order problem.

This work leverages existing manuscripts on fractional boundary value problems that utilize Krasnosel'skii's Fixed Point Theorem. A wide array of fixed point theorems have been utilized to establish the existence or nonexistence of positive solutions for similar problems, as seen in [8–15]. In this work, we posit parameter constraints on λ formulated in terms of the liminf and limsup of the nonlinearity f. The inherited Green's function properties are critically important to the implementation of the Fixed Point Theorem. For further reading of recent work on proving the existence of solutions for fractional boundary value problems, we refer the reader to [16–19].

The remainder of the work is organized as follows. In section two, we introduce key definitions related to the Riemann-Liouville fractional derivative and present Krasnosel'skii's Fixed Point Theorem. The next two sections focus on constructing the Green's function using convolution and induction and proving crucial properties. In sections five and six, we establish existence and nonexistence results based upon the parameter λ. To conclude, we present an example.

2. Preliminaries and the Fixed Point Theorem

We begin with definitions of the Riemann-Liouville fractional integral and derivative. We refer to [20–23] for further study of fractional calculus and fractional differential equations.

Definition 1. *Let $\nu > 0$. The Riemann-Liouville fractional integral of a function u of order ν, denoted $I^{\nu}_{0+}u$, is defined as*

$$I^{\nu}_{0+}u(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}u(s)ds,$$

provided the right-hand side exists.

Definition 2. *Let n denote a positive integer and assume $n-1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order α of the function $u : [0,1] \to \mathbb{R}$, denoted $D^{\alpha}_{0+}u$, is defined as*

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds = D^n I^{n-\alpha}_{0+}u(t),$$

provided the right-hand side exists.

Now, we present Krasnosel'skii's Fixed Point Theorem.

Theorem 1 (Krasnosel'skii Fixed Point Theorem)**.** *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset X$ be a cone in $\mathcal{P}$. Assume that $\Omega_1,\ \Omega_2$ are open sets with $0 \in \Omega_1$, and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$ be a completely continuous operator such that either*

1) $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2$; *or*

2) $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2.$

Then, T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. The Green's function

Now, we construct the Green's function used for Equations (1) and (2) by utilizing induction with the convolution of a lower-order problem and a left-focal problem. The procedure is similar to that found in [3]. We include it here for completeness and note that all integrals herein are Riemann.

The Green's function for the left-focal boundary value problem

$$-u'' = 0, \quad 0 < t < 1, \quad u'(0) = 0, \quad u(1) = 0,$$

is given by

$$G_{left}(t,s) = \begin{cases} 1 - s, & 0 \leq s < t \leq 1, \\ 1 - t, & 0 \leq t < s \leq 1. \end{cases}$$

The Green's function for the Riemann-Liouville fractional two-point boundary value problem

$$-D^{\alpha}_{0^+} u = 0, \quad 0 < t < 1, \quad u^{(i)}(0) = 0,\ i = 0, 1, \ldots, m-2, \quad D^{\beta}_{0^+} u(1) = 0$$

is given by ([24])

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1-\beta} - (t-s)^{\alpha-1}, & 0 \leq s < t \leq 1, \\ \\ t^{\alpha-1}(1-s)^{\alpha-1-\beta}, & 0 \leq t \leq s < 1. \end{cases}$$

For $k = 1, \ldots, n-1$, recursively define $G_k(t,s)$ by

$$G_k(t,s) = -\int_0^1 G_{k-1}(t,r) G_{left}(r,s)\,dr.$$

Then,

$$G_n(t,s) = -\int_0^1 G_{n-1}(t,r) G_{left}(r,s)\,dr \tag{3}$$

is the Green's function for

$$-D^{\alpha+2n}_{0^+} u(t) = 0, \quad 0 < t < 1,$$

with boundary conditions (2), and $G_{n-1}(t,s)$ is the Green's function for

$$-D_{0^+}^{\alpha+2(n-1)}u(t)=0,\quad 0<t<1,$$

with boundary conditions

$$u^{(i)}(0)=0,\ i=0,1,\ldots,m-2,\quad D_{0^+}^{\beta}u(1)=0,$$

$$D_{0^+}^{\alpha+2l+1}u(0)=D_{0^+}^{\alpha+2l}u(1)=0,\ l=0,1,\ldots,n-2.$$

We proceed with induction. For the base case $k=1$, consider the linear differential equation

$$D_{0^+}^{\alpha+2}u(t)+h(t)=0,\quad 0<t<1,$$

satisfying the boundary conditions

$$u^{(i)}(0)=0,\ i=0,1,\ldots,m-2,\quad D_{0^+}^{\beta}u(1)=0,$$

$$D_{0^+}^{\alpha+1}u(0)=0,\quad D_{0^+}^{\alpha}u(1)=0.$$

Now, we make a change of variable

$$v(t)=D_{0^+}^{\alpha+2-2}u(t),$$

so that

$$D^2v(t)=D^2D_{0^+}^{\alpha+2-2}u(t)=D_{0^+}^{\alpha+2}u(t)=-h(t).$$

Since $v(t)=D_{0^+}^{\alpha}u(t)$,

$$v'(0)=D_{0^+}^{\alpha+1}u(0)=0\quad\text{and}\quad v(1)=D_{0^+}^{\alpha}u(1)=0.$$

Therefore, v satisfies the left-focal boundary value problem

$$v''+h(t)=0,\quad 0<t<1,$$

$$v'(0)=0,\quad v(1)=0.$$

The solution of this boundary value problem is

$$v(t)=\int_0^1 G_{left}(t,s)h(s)ds.$$

Additionally, u now satisfies a lower-order boundary value problem,

$$D_{0^+}^{\alpha}u(t)=v(t),\quad 0<t<1,$$

$$u^{(i)}(0)=0,\ i=0,1,\ldots,m-2,\quad D_{0^+}^{\beta}u(1)=0.$$

The solution of the lower-order fractional boundary value problem is

$$\begin{aligned} u(t) &= \int_0^1 G_0(t,s)(-v(s))ds \\ &= \int_0^1 G_0(t,s)\left(-\int_0^1 G_{left}(s,r)h(r)ds\right)dr \\ &= \int_0^1 \left(\int_0^1 -G_0(t,s)G_{left}(s,r)ds\right)h(r)dr. \end{aligned}$$

Therefore,

$$u(t) = \int_0^1 G_1(t,s)h(s)ds,$$

where

$$G_1(t,s) = -\int_0^1 G_0(t,r)G_{left}(r,s)dr.$$

For the inductive step, the argument is similar. Assume that $k = n-1$ is true, and consider the linear differential equation

$$D_{0^+}^{\alpha+2n}u(t) + k(t) = 0, \quad 0 < t < 1,$$

satisfying boundary conditions (2).

We make a similar change of variable

$$v(t) = D_{0^+}^{\alpha+2(n-1)}u(t) = D_{0^+}^{\alpha+2n-2}u(t),$$

so that

$$D^2 v(t) = D_{0^+}^{\alpha+2n} = -k(t),$$

and

$$v'(0) = D_{0^+}^{\alpha+2(n-1)+1}u(0) = 0 \quad \text{and} \quad v(1) = D_{0^+}^{\alpha+2(n-1)}v(1) = 0.$$

Thus, $v(t)$ satisfies the left-focal boundary value problem

$$v'' + k(t) = 0, \quad 0 < t < 1,$$

$$v'(0) = 0, \quad v(1) = 0,$$

while $u(t)$ satisfies a lower-order problem

$$D_{0^+}^{\alpha+2(n-1)}u(t) = v(t), \quad 0 < t < 1,$$

$$u(0) = 0, \quad D_{0^+}^{\beta}u(1) = 0,$$

$$D_{0^+}^{\alpha+2l+1}u(0) = D_{0^+}^{\alpha+2l}u(1) = 0,\ l = 0, 1, \dots, n-2.$$

By induction,

$$
\begin{aligned}
u(t) &= \int_0^1 G_{n-1}(t,s)(-v(s))ds \\
&= \int_0^1 \left(-\int_0^1 G_{n-1}(t,s)G_{left}(s,r)ds\right)k(r)dr \\
&= \int_0^1 G_n(t,s)k(s)ds.
\end{aligned}
$$

Therefore,

$$u(t) = \int_0^1 G_n(t,s)k(s)ds,$$

where

$$G_n(t,s) = -\int_0^1 G_{n-1}(t,r)G_{left}(r,s)dr.$$

So, the unique solution to

$$D_{0^+}^{\alpha+2n}u(t) + k(t) = 0, \quad 0 < t < 1,$$

satisfying boundary conditions (2) is given by

$$u(t) = \int_0^1 G_n(t,s)k(s)ds.$$

4. Green's function properties

We now discuss properties for $G_n(t,s)$ that are inherited from $G_0(t,s)$ and $G_{left}(t,s)$. The results of the first lemma regarding $G_{left}(t,s)$ are easily verifiable.

Lemma 1. *For* $(t,s) \in [0,1] \times [0,1]$, $G_{left}(t,s) \in C^{(1)}$ *and* $G_{left}(t,s) \geq 0$.

The following lemma regarding $G_0(t,s)$ is Lemma 3.1 proved in [2].

Lemma 2.

1) *For* $(t,s) \in [0,1] \times [0,1)$, $G_0(t,s) \in C^{(1)}$.

2) *For* $(t,s) \in (0,1) \times (0,1)$, $G_0(t,s) > 0$ *and* $\dfrac{\partial}{\partial t}G_0(t,s) > 0$.

3) *For* $(t,s) \in [0,1] \times [0,1)$, $t^{\alpha-1}G_0(1,s) \leq G_0(t,s) \leq G_0(1,s)$.

Finally, we prove inherited properties for $G_n(t,s)$ from Lemma 2.

Lemma 3.

1) *For* $(t,s) \in [0,1] \times [0,1)$, $G_n(t,s) \in C^{(1)}$.

2) *For* $(t,s) \in (0,1) \times (0,1)$, $(-1)^n G_n(t,s) > 0$ *and* $(-1)^n \dfrac{\partial}{\partial t}G_n(t,s) > 0$.

3) *For* $(t,s) \in [0,1] \times [0,1)$,

$$(-1)^n t^{\alpha-1}G_n(1,s) \leq (-1)^n G_n(t,s) \leq (-1)^n G_n(1,s).$$

Proof. We proceed inductively for each part.

For 1) with $(t,s)\in[0,1]\times[0,1)$, we begin with the base case $k=1$:

$$G_1(t,s)=-\int_0^1 G_0(t,r)G_{left}(r,s)ds.$$

By Lemmas 1 and 2, $G_1(t,s)\in C^{(1)}$.

Assume that $k=n-1$ is true. Then, from Equation (3),

$$G_n(t,s)=-\int_0^1 G_{n-1}(t,r)G_{left}(r,s)ds.$$

By induction and Lemma 1 1), $G_n(t,s)\in C^{(1)}$.

For 2) with $(t,s)\in(0,1)\times(0,1)$ and using Lemmas 1 and 2 2), we begin with the base case $k=1$:

$$(-1)^1G_1(t,s)=-\left(-\int_0^1 G_0(t,r)G_{left}(r,s)dr\right)>0$$

and

$$(-1)^1\frac{\partial}{\partial t}G_1(t,s)=-\left(-\int_0^1 \frac{\partial}{\partial t}G_0(t,r)G_{left}(r,s)dr\right)>0.$$

Assume that $k=n-1$ is true. Then, from Equation (3) and by induction and Lemma 1,

$$\begin{aligned}(-1)^nG_n(t,s)&=(-1)^n\left(-\int_0^1 G_{n-1}(t,r)G_{left}(r,s)dr\right)\\&=(-1)^2\left(\int_0^1(-1)^{n-1}G_{n-1}(t,r)G_{left}(r,s)dr\right)\\&>0,\end{aligned}$$

and

$$\begin{aligned}(-1)^n\frac{\partial}{\partial t}G_n(t,s)&=(-1)^n\left(-\int_0^1 \frac{\partial}{\partial t}G_{n-1}(t,r)G_{left}(r,s)dr\right)\\&=(-1)^2\left(\int_0^1(-1)^{n-1}\frac{\partial}{\partial t}G_{n-1}(t,r)G_{left}(r,s)dr\right)\\&>0.\end{aligned}$$

For 3) with $(t,s)\in[0,1]\times[0,1)$ and using Lemma 2 3), we begin with the base case $k=1$:

$$\begin{aligned}(-1)^1t^{\alpha-1}G_1(1,s)&=-t^{\alpha-1}\left(-\int_0^1 G_0(1,r)G_{left}(r,s)dr\right)\\&=-\left(\int_0^1 -t^{\alpha-1}G_0(1,r)G_{left}(r,s)dr\right)\\&\le-\left(\int_0^1 -G_0(t,r)G_{left}(r,s)dr\right)\\&=-\left(-\int_0^1 G_0(t,r)G_{left}(r,s)dr\right)\end{aligned}$$

$$= (-1)^1 G_1(t,s),$$

and

$$\begin{aligned}
(-1)^1 G_1(t,s) &= -\left(-\int_0^1 G_0(t,r)G_{left}(r,s)dr\right) \\
&= \int_0^1 G_0(t,r)G_{left}(r,s)dr \\
&\leq \int_0^1 G_0(1,r)G_{left}(r,s)dr \\
&= -\left(-\int_0^1 G_0(1,r)G_{left}(r,s)dr\right) \\
&= (-1)^1 G_1(1,s).
\end{aligned}$$

Assume that $k = n-1$ is true. Then, from Equation (3),

$$\begin{aligned}
(-1)^n t^{\alpha-1} G_n(1,s) &= (-1)^n t^{\alpha-1}\left(-\int_0^1 G_{n-1}(1,r)G_{left}(r,s)dr\right) \\
&= (-1)^2\left(\int_0^1 (-1)^{n-1} t^{\alpha-1} G_{n-1}(1,r)G_{left}(r,s)dr\right) \\
&\leq (-1)^2\left(\int_0^1 (-1)^{n-1} G_{n-1}(t,r)G_{left}(r,s)dr\right) \\
&= (-1)^n\left(-\int_0^1 G_{n-1}(t,r)G_{left}(r,s)dr\right) \\
&= (-1)^n G_n(t,s),
\end{aligned}$$

and

$$\begin{aligned}
(-1)^n G_n(t,s) &= (-1)^n\left(-\int_0^1 G_{n-1}(t,r)G_{left}(r,s)dr\right) \\
&= (-1)^2\left(\int_0^1 (-1)^{n-1} G_{n-1}(t,r)G_{left}(r,s)dr\right) \\
&\leq (-1)^2\left(\int_0^1 (-1)^{n-1} G_{n-1}(1,r)G_{left}(r,s)dr\right) \\
&= (-1)^n\left(-\int_0^1 G_{n-1}(1,r)G_{left}(r,s)dr\right) \\
&= (-1)^n G_n(1,s).
\end{aligned}$$

□

5. Existence of solutions

With the Green's function established and necessary properties proved, we now turn our attention to the existence of positive solutions to Equations (1) and (2) based upon the parameter λ using Krasnosel'skii Fixed Point Theorem.

Define the constants

$$A_{G_n} = \int_0^1 (-1)^n s^{\alpha-1} G_n(1,s)g(s)ds, \quad B_{G_n} = \int_0^1 (-1)^n G_n(1,s)g(s)ds,$$

$$F_0 = \limsup_{u\to 0^+} \frac{f(u)}{u}, \quad f_0 = \liminf_{u\to 0^+} \frac{f(u)}{u},$$
$$F_\infty = \limsup_{u\to \infty} \frac{f(u)}{u}, \quad f_\infty = \liminf_{u\to \infty} \frac{f(u)}{u}.$$

Let $\mathcal{B} = C[0,1]$ be a Banach space with norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Define the cone

$$\begin{aligned}\mathcal{P} = \{&u \in \mathcal{B} : u(0) = 0,\ u(t) \text{ is nondecreasing, and} \\ &t^{\alpha-1}u(1) \le u(t) \le u(1) \text{ on } [0,1]\big\}.\end{aligned}$$

The inequality condition $\mathcal{P}$ is an inherited result from $G_n(t,s)$ in Lemma 3.

Define the operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Tu(t) = (-1)^n \lambda \int_0^1 G_n(t,s) g(s) f(u(s)) ds.$$

Lemma 4. *The operator $T : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. Let $u \in \mathcal{P}$. Then, by definition,

$$Tu(0) = (-1)^n \lambda \int_0^1 G_n(0,s) g(s) f(u(s)) ds = 0.$$

Also, for $t \in (0,1)$ and by Lemma 3 2),

$$\frac{\partial}{\partial t}[Tu(t)] = (-1)^n \lambda \int_0^1 \frac{\partial}{\partial t} G_n(t,s) g(s) f(u(s)) ds > 0,$$

which implies that $Tu(t)$ is nondecreasing.

Next, for $t \in [0,1]$ and by Lemma 3,

$$\begin{aligned} t^{\alpha-1} Tu(1) &= t^{\alpha-1}(-1)^n \lambda \int_0^1 G_n(1,s) g(s) f(u(s)) ds \\ &\le (-1)^n \lambda \int_0^1 G_n(t,s) g(s) f(u(s)) ds \\ &= Tu(t), \end{aligned}$$

and

$$\begin{aligned} Tu(t) &= (-1)^n \lambda \int_0^1 G_n(t,s) g(s) f(u(s)) ds \\ &\le (-1)^n \lambda \int_0^1 G_n(1,s) g(s) f(u(s)) ds \\ &= Tu(1). \end{aligned}$$

Therefore, $Tu \in \mathcal{P}$, and by the Arzela-Ascoli Theorem, T is completely continuous. □

Theorem 2. *If*

$$\frac{1}{A_{G_n} f_\infty} < \lambda < \frac{1}{B_{G_n} F_0},$$

then (1)*,* (2) *has at least one positive solution.*

Proof. Since $F_0 \lambda B_{G_n} < 1$, there exists an $\epsilon > 0$ such that

$$(F_0 + \epsilon)\lambda B_{G_n} \leq 1.$$

Also since

$$F_0 = \limsup_{u \to 0^+} \frac{f(u)}{u},$$

there exists an $H_1 > 0$ such that

$$f(u) \leq (F_0 + \epsilon)u \quad \text{for} \quad u \in (0, H_1].$$

Define $\Omega_1 = \{u \in \mathcal{B} : \|u\| < H_1\}$. If $u \in \mathcal{P} \cap \partial\Omega_1$, then $\|u\| = H_1$, and

$$\begin{aligned} |(Tu)(1)| &= (-1)^n \lambda \int_0^1 G_n(1,s) g(s) f(u(s)) ds \\ &\leq (-1)^n \lambda \int_0^1 G_n(1,s) g(s)(F_0 + \epsilon) u(s) ds \\ &\leq (F_0 + \epsilon) u(1) \lambda \int_0^1 (-1)^n G_n(1,s) g(s) ds \\ &\leq (F_0 + \epsilon)\|u\| \lambda B_{G_n} \\ &\leq \|u\|. \end{aligned}$$

Since $Tu \in \mathcal{P}$, $\|Tu\| \leq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$.

Next, since $f_\infty \lambda > \dfrac{1}{A_{G_n}}$, there exists a $c \in (0,1)$ and an $\epsilon > 0$ such that

$$(f_\infty - \epsilon)\lambda > \left((-1)^n \int_c^1 s^{\alpha-1} G_n(1,s) g(s) ds \right)^{-1}.$$

Since

$$f_\infty = \liminf_{u \to \infty} \frac{f(u)}{u},$$

there exists an $H_3 > 0$ such that

$$f(u) \geq (f_\infty - \epsilon)u \quad \text{for} \quad u \in [H_3, \infty).$$

Define

$$H_2 = \max\left\{ \frac{H_3}{c^{\alpha-1}}, 2H_1 \right\},$$

and define $\Omega_2 = \{u \in \mathcal{B} : \|u\| < H_2\}$.

Let $u \in \mathcal{P} \cap \partial\Omega_2$. Then, $\|u\| = H_2$. Notice for $t \in [c, 1]$,

$$u(t) \geq t^{\alpha-1}u(1) \geq c^{\alpha-1}H_2 \geq c^{\alpha-1}\frac{H_3}{c^{\alpha-1}} = H_3.$$

Therefore,

$$\begin{aligned}
|(Tu)(1)| &\geq (-1)^n\lambda \int_c^1 G_n(1,s)g(s)f(u(s))ds \\
&\geq \lambda \int_c^1 (-1)^n G_n(1,s)g(s)(f_\infty - \epsilon)u(s)ds \\
&\geq \lambda(f_\infty - \epsilon)u(1)(-1)^n \int_c^1 s^{\alpha-1}G_n(1,s)g(s)ds \\
&\geq \|u\|.
\end{aligned}$$

Hence, $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$. Notice since $H_1 < H_2$ we have $\overline{\Omega}_1 \subset \Omega_2$. Thus, by Theorem 1 1), T has a fixed point $u \in \mathcal{P}$. By the definition of T, this fixed point is a positive solution of Equations (1) and (2). □

Theorem 3. *If*

$$\frac{1}{A_{G_n}f_0} < \lambda < \frac{1}{B_{G_n}F_\infty},$$

then (1)*,* (2) *has at least one positive solution.*

Proof. Since $f_0\lambda A_{G_n} > 1$, there exists an $\epsilon > 0$ such that

$$(f_0 - \epsilon)\lambda A_{G_n} \geq 1.$$

Then, since

$$f_0 = \liminf_{u \to 0^+} \frac{f(u)}{u},$$

there exists an $H_1 > 0$ such that

$$f(u) \geq (f_0 - \epsilon)u, \quad t \in (0, H_1].$$

Define $\Omega_1 = \{u \in \mathcal{B} : \|u\| < H_1\}$. If $u \in \mathcal{P} \cap \partial\Omega_1$, then $u(t) \leq H_1$ for $t \in [0, 1]$. So,

$$\begin{aligned}
|(Tu)(1)| &= (-1)^n\lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\
&\geq (-1)^n\lambda \int_0^1 G_n(1,s)g(s)(f_0 - \epsilon)u(s)ds \\
&\geq \lambda(f_0 - \epsilon)u(1) \int_0^1 (-1)^n s^{\alpha-1}G_n(1,s)g(s)ds \\
&\geq \lambda(f_0 - \epsilon)\|u\|A_{G_n} \\
&\geq \|u\|.
\end{aligned}$$

Thus, $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$.

Next, since $F_\infty B_{G_n}\lambda < 1$, there exists an $\epsilon \in (0,1)$ such that

$$((F_\infty + \epsilon)B_{G_n} + \epsilon)\lambda \leq 1.$$

Since

$$F_\infty = \limsup_{u\to\infty} \frac{f(u)}{u},$$

there exists an $H_3 > 0$ such that

$$f(u) \leq (F_\infty + \epsilon)u, \quad u \in [H_3, \infty).$$

Define

$$M = \max_{u\in[0,H_3]} f(u).$$

Now, there exists a $k \in (0,1)$ with

$$(-1)^n \int_0^k G_n(1,s)g(s)ds \leq \frac{\epsilon}{M}.$$

Let

$$H_2 = \max\left\{2H_1, \frac{H_3}{k^{\alpha-1}}, 1\right\},$$

and define $\Omega_2 = \{u \in \mathcal{B} : \|u\| < H_2\}$. Let $u \in \mathcal{P} \cap \partial\Omega_2$. Then, $\|u\| = H_2$ and so,

$$u(1) = H_2 \geq \frac{H_3}{k^{\alpha-1}} > H_3.$$

Now, $u(0) = 0$. So, by the Intermediate Value Theorem, there exists a $\gamma \in (0,1)$ with $u(\gamma) = H_3$. But, for $t \in [k,1]$, we have

$$u(t) \geq t^{\alpha-1}u(1) = t^{\alpha-1}H_2 \geq k^{\alpha-1}\frac{H_3}{k^{\alpha-1}} = H_3.$$

So, $\gamma \in (0,k]$. Moreover, since $u(t)$ is nondecreasing, this implies

$$0 \leq u(t) \leq H_3, \quad t \in [0,\gamma),$$

and

$$u(t) \geq H_3, \quad t \in (\gamma, 1].$$

Therefore,

$$\begin{aligned}
|(Tu)(1)| &= (-1)^n\lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\
&= \lambda\left((-1)^n \int_0^\gamma G_n(1,s)g(s)f(u(s))ds \,+\, (-1)^n \int_\gamma^1 G_n(1,s)g(s)f(u(s))ds\right) \\
&\leq \lambda\left(M \int_0^\gamma (-1)^n G_n(1,s)g(s)ds \,+\, (-1)^n \int_\gamma^1 G_n(1,s)g(s)(F_\infty + \epsilon)u(s)ds\right)
\end{aligned}$$

$$\leq \lambda \left(M\frac{\epsilon}{M} + (F_\infty + \epsilon)u(1) \int_\gamma^1 (-1)^n G_n(1,s)g(s)ds \right)$$
$$\leq \lambda(\epsilon + (F_\infty + \epsilon)\|u\|\, B_{G_n})$$
$$\leq \lambda(\epsilon\|u\| + (F_\infty + \epsilon)\|u\|\, B_{G_n})$$
$$= \lambda\|u\|(\epsilon + (F_\infty + \epsilon)B_{G_n})$$
$$\leq \|u\|$$

Thus, $\|Tu\| \leq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$. Notice that since $H_1 < H_2$ we have $\overline{\Omega}_1 \subset \Omega_2$. Thus, by Theorem 1 2), T has a fixed point $u \in \mathcal{P}$. By the definition of T, this fixed point is a positive solution of Equations (1) and (2). □

Remark 1. *Since our interval is* $[0,1]$*, it is clear that* $A_{G_n} < B_{G_n}$*. Thus, the size differential between* f_∞ *and* F_0 *for Theorem 2 and* f_0 *and* F_∞ *for Theorem 3. To further expand, for Theorem 2,* $F_0 < A_{G_n} f_\infty / B_{G_n}$ *to ensure the existence of at least one positive solution. Since* A_{G_n} *and* B_{G_n} *are independent of* f*, this is easily manageable.*

6. Nonexistence results

Next, we provide nonexistence of positive solution results based on the size of the parameter λ. First, we need the following lemma.

Lemma 5. *Suppose* $D_{0^+}^{\alpha+2n}u \in C[0,1]$*. If* $(-1)^n(-D_{0^+}^{\alpha+2n}u(t)) \geq 0$ *for all* $t \in [0,1]$ *and* $u(t)$ *satisfies* (2)*, then*

1) $u'(t) \geq 0, \quad 0 \leq t \leq 1$*, and*

2) $t^{\alpha-1}u(1) \leq u(t) \leq u(1), \quad 0 \leq t \leq 1.$

Proof. Let $0 \leq t \leq 1$.

For 1), by Lemma 3 2),

$$\begin{aligned} u'(t) &= \int_0^1 \frac{\partial}{\partial t} G_n(t,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\ &= \int_0^1 (-1)^n \frac{\partial}{\partial t} G_n(t,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\ &> 0. \end{aligned}$$

For 2), by Lemma 3 3),

$$\begin{aligned} t^{\alpha-1}u(1) &= t^{\alpha-1}\int_0^1 G_n(1,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\ &= \int_0^1 (-1)^n t^{\alpha-1} G_n(1,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\ &\leq \int_0^1 (-1)^n G_n(t,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\ &= \int_0^1 G_n(t,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\ &= u(t), \end{aligned}$$

and

$$\begin{aligned}
u(t) &= \int_0^1 G_n(t,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 (-1)^n G_n(t,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&\le \int_0^1 (-1)^n G_n(1,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 G_n(1,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= u(1).
\end{aligned}$$

□

Theorem 4. *If*

$$\lambda < \frac{u}{B_{G_n} f(u)},$$

for all $u \in (0,\infty)$*, then no positive solution exists to Equations* (1) *and* (2)*.*

Proof. For contradiction, suppose that $u(t)$ is a positive solution to Equations (1) and (2). Then, $(-1)^n(-D_{0^+}^{\alpha+2n}u(t)) = \lambda g(t) f(u(t)) \ge 0$. So by Lemma 5,

$$\begin{aligned}
u(1) &= (-1)^n \lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\
&< (-1)^n (B_{G_n})^{-1} \int_0^1 G_n(1,s)g(s)u(s)ds \\
&\le u(1)(B_{G_n})^{-1} \int_0^1 (-1)^n G_n(1,s)g(s)ds \\
&= u(1),
\end{aligned}$$

a contradiction. □

Theorem 5. *If*

$$\lambda > \frac{u}{A_{G_n} f(u)}$$

for all $u \in (0,\infty)$*, then no positive solution exists to Equations* (1) *and* (2)*.*

Proof. For contradiction, suppose that $u(t)$ is a positive solution to Equations (1) and (2). Then, $(-1)^n(-D_{0^+}^{\alpha+2n}u(t)) = \lambda g(t) f(u(t)) \ge 0$. So by Lemma 5,

$$\begin{aligned}
u(1) &= (-1)^n \lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\
&> (-1)^n (A_{G_n})^{-1} \int_0^1 G_n(1,s)g(s)u(s)ds \\
&\ge u(1)(A_{G_n})^{-1} \int_0^1 (-1)^n s^{\alpha-1} G_n(1,s)g(s)ds \\
&= u(1),
\end{aligned}$$

a contradiction. □

7. Examples

Finally, we discuss examples to demonstrate a way to use these theorems and ensure positive solutions exist for fractional boundary value problems with meaningful λ values.

Let $n = 2$, $g(t) = t$, and $f(u) = u(10u+1)/(u+1)$. We note that g and f satisfy their respective conditions. We see that $f(u)/u = (10u + 1)/(u + 1)$ and find

$$F_\infty = f_\infty = 10 \quad \text{and} \quad F_0 = f_0 = 1.$$

The fractional Green's function with α and β parameters is

$$G_0(1, s; \alpha, \beta) = \frac{(1 - s)^{\alpha-1-\beta} - (1 - s)^{\alpha-1}}{\Gamma(2.5)}.$$

Additionally, we have in terms of α and β

$$A_{G_2}(\alpha, \beta) = \int_0^1 \int_0^1 \int_0^1 s^{\alpha-1} G_0(1, r_2; \alpha, \beta) G_{left}(r_2, r_1) G_{left}(r_1, s) s dr_2 dr_1 ds,$$

and

$$B_{G_2}(\alpha, \beta) = \int_0^1 \int_0^1 \int_0^1 G_0(1, r_2; \alpha, \beta) G_{left}(r_2, r_1) G_{left}(r_1, s) s dr_2 dr_1 ds.$$

A closed form for these constants in terms of the parameters α and β would be difficult to find. Instead, we evaluate sample values for these in our examples.

Example 1. *Set* $m = 3$, $\alpha = 2.5$, *and* $\beta = 1.5$ *so that* $A_{G_2}(2.5, 1.5) \approx 0.047458$ *and* $B_{G_2}(2.5, 1.5) \approx 0.087190$. *We have*

$$\frac{1}{A_{G_2} f_\infty} \approx \frac{1}{0.047458 \cdot 10} \approx 2.107,$$

and

$$\frac{1}{B_{G_2} F_0} \approx \frac{1}{0.087190 \cdot 1} \approx 11.469.$$

Additionally, we calculate

$$\inf_{u \in (0,\infty)} \frac{u}{B_{G_2} f(u)} \approx \frac{1}{0.087190} \cdot \frac{1}{10} \approx 1.147,$$

and

$$\sup_{u \in (0,\infty)} \frac{u}{A_{G_2} f(u)} = \frac{1}{0.047458} \cdot 1 \approx 21.071.$$

The fractional boundary value problem is

$$D_{0^+}^{6.5} u(t) + \lambda t \frac{u(10u + 1)}{u + 1} = 0, \quad 0 < t < 1,$$

subject to

$$u(0) = u'(0) = 0, \quad D_{0^+}^{1.5}(1) = 0,$$
$$D_{0^+}^{3.5}(0) = D_{0^+}^{2.5}(1) = 0, \quad D_{0^+}^{5.5}(0) = D_{0^+}^{4.5}(1) = 0.$$

Applying Theorem 2, we find that a positive solution exists if $2.107 < \lambda < 11.469$. *Applying Theorems 4 and 5, we find that a positive solution does not exist if* $0 < \lambda < 1.147$ *or* $\lambda > 21.071$.

Example 2. *Set* $m = 4$, $\alpha = \pi$, *and* $\beta = e$ *so that* $A_{G_2}(\pi, e) \approx 0.076124$ *and* $B_{G_2}(\pi, e) \approx 0.165625$. *We have*

$$\frac{1}{A_{G_2} f_\infty} \approx \frac{1}{0.076124 \cdot 10} \approx 1.314,$$

and

$$\frac{1}{B_{G_2} F_0} \approx \frac{1}{0.165625 \cdot 1} \approx 6.038.$$

Additionally, we calculate

$$\inf_{u \in (0,\infty)} \frac{u}{B_{G_2} f(u)} \approx \frac{1}{0.165625} \cdot \frac{1}{10} \approx 0.604,$$

and

$$\sup_{u \in (0,\infty)} \frac{u}{A_{G_2} f(u)} = \frac{1}{0.076124} \cdot 1 \approx 13.136.$$

The fractional boundary value problem is

$$D_{0^+}^{\pi+4} u(t) + \lambda t \frac{u(10u+1)}{u+1} = 0, \quad 0 < t < 1,$$

subject to

$$u(0) = u'(0) = u''(0) = 0, \quad D_{0^+}^{e}(1) = 0,$$
$$D_{0^+}^{\pi+1}(0) = D_{0^+}^{\pi}(1) = 0, \quad D_{0^+}^{\pi+3}(0) = D_{0^+}^{\pi+2}(1) = 0.$$

Applying Theorem 2, we find that a positive solution exists if $1.314 < \lambda < 6.038$. *Applying Theorems 4 and 5, we find that a positive solution does not exist if* $0 < \lambda < 0.604$ *or* $\lambda > 13.136$.

Example 3. *Set* $m = 10$, $\alpha = 9.75$, *and* $\beta = 3.25$ *so that* $A_{G_2}(9.75, 3.25) \approx 9.794191061019805 \times 10^{-9}$ *and* $B_{G_2}(9.75, 3.25) \approx 6.010879852040359 \times 10^{-8}$. *We have*

$$\frac{1}{A_{G_2} f_\infty} \approx \frac{1}{9.794191061019805 \times 10^{-9} \cdot 10} \approx 10{,}210{,}134,$$

and

$$\frac{1}{B_{G_2} F_0} \approx \frac{1}{6.010879852040359 \times 10^{-8} \cdot 1} \approx 16{,}636{,}500.$$

Additionally, we calculate

$$\inf_{u\in(0,\infty)} \frac{u}{B_{G_2}f(u)} \approx \frac{1}{6.010879852040359 \times 10^{-8}} \cdot \frac{1}{10} \approx 1,663,650,$$

and

$$\sup_{u\in(0,\infty)} \frac{u}{A_{G_2}f(u)} = \frac{1}{9.794191061019805 \times 10^{-9}} \cdot 1 \approx 102,101,337.$$

The fractional boundary value problem is

$$D_{0^+}^{13.75}u(t) + \lambda t\frac{u(10u+1)}{u+1} = 0, \quad 0 < t < 1,$$

subject to

$$u^{(i)}(0) = 0,\ 0 \le i \le 8, \quad D_{0^+}^{3.25}(1) = 0,$$
$$D_{0^+}^{10.75}(0) = D_{0^+}^{9.75}(1) = 0, \quad D_{0^+}^{12.75}(0) = D_{0^+}^{11.75}(1) = 0.$$

Applying Theorem 2, we find that a positive solution exists if $10,210,134 < \lambda < 16,636,500$*. Applying Theorems 4 and 5, we find that a positive solution does not exist if* $0 < \lambda < 1,663,650$ *or* $\lambda > 102,101,337$.

Institutional review board statement: Not applicable.

Informed consent statement: Not applicable.

Conflict of interest: The author declares no conflict of interest.

References

1. Eloe PW, Neugebauer JT. Convolutions and Green's functions for two families of boundary value problems for fractional differential equations. Electronic Journal of Differential Equations. 2016; 2016: 1–13.
2. Lyons JW, Neugebauer JT. Two point fractional boundary value problems with a fractional boundary condition. Fractional Calculus and Applied Analysis. 2018; 21(2): 442–461.
3. Neugebauer JT, Wingo AG. Positive solutions for a fractional boundary value problem with lidstone-like boundary conditions. Kragujevac Journal of Mathematics. 2024; 48(2): 309–322.
4. Graef JR, Qian C, Yang B. A three point boundary value problem for nonlinear fourth order differential equations. Journal of Mathematical Analysis and Applications. 2003; 287(1): 217–233.
5. Graef JR, Yang B. Existence and nonexistence of positive solutions of fourth order nonlinear boundary value problems. Applicable Analysis. 2000; 74(1–2): 201–214.
6. Graef JR, Yang B. Positive solutions to a multi-point higher order boundary-value problem. Journal of Mathematical Analysis and Applications. 2006; 316(2): 409–421.
7. Graef JR, Henderson J, Yang B. Positive solutions of a nonlinear higher order boundary-value problem. Electronic Journal of Differential Equations. 2007; 2007(45): 1–10.
8. Agarwal RP, O'Regan D, Staněk S. Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. Journal of Mathematical Analysis and Applications. 2010; 371(1): 57–68. doi: 10.1016/j.jmaa.2010.04.034
9. Bai Z, Lü H. Positive solutions for boundary value problem of nonlinear fractional differential equation. Journal of Mathematical Analysis and Applications 2005; 311(2): 495–505. doi: 10.1016/j.jmaa.2005.02.052

10. Dimitrov ND, Jonnalagadda JM. Existence and nonexistence results for a fourth-order boundary value problem with sign-changing Green's function. Mathematics. 2024; 12(16): 1–12.
11. Henderson J, Luca R. Existence of positive solutions for a singular fraction boundary value problem. Nonlinear Analysis: Modelling and Control. 2017; 22(1): 99–114.
12. Kaufmann ER, Mboumi E. Positive solutions of a boundary value problem for a nonlinear fractional differential equation. Electronic Journal of Qualitative Theory of Differential Equations. 2008; 2008(3): 1–11.
13. Rehman MU, Khan RA, Eloe PW. Positive solutions of nonlocal boundary value problem for higher order fractional differential system. Dynamic Systems and Applications. 2011; 20: 169–182.
14. Yang B. Upper estimate for positive solutions of the $(p, n-p)$ conjugate boundary value problem. Journal of Mathematical Analysis and Applications. 2012; 390(2): 535–548. doi: 10.1016/j.jmaa.2012.01.054
15. Youssri YH, Atta AG. Chebyshev Petrov–Galerkin method for nonlinear time-fractional integro-differential equations with a mildly singular kernel. Journal of Applied Mathematics and Computing. 2025; 1–21. doi: 10.1007/s12190-025-02371-w
16. Albeladi G, Gamal M, Youssri YH. G-Metric spaces via fixed point techniques for Ψ-contraction with applications. Fractal and Fractional. 2025; 9(3): 1–31. doi: 10.3390/fractalfract9030196
17. Lyons JW, Neugebauer JT, Wingo AG. Existence and nonexistence of positive solutions for fractional boundary value problems with Lidstone-inspired fractional conditions. Mathematics. 2025; 13(8): 1–15.
18. Taema MA, Zeen El-Deen MR. Youssri YH. Fourth-kind Chebyshev operational tau algorithm for fractional Bagley-Torvik equation. Frontiers in Scientific Research and Technology. 2025; 11(1): 1–8.
19. Youssri YH, Alnaser LA, Atta AG. A spectral collocation approach for time-fractional Korteweg-deVries-Burgers equation via first-kind Chebyshev polynomials. Contemp. Math. 2025; 6(2): 1501–1519. doi: 10.37256/cm.6220255948
20. Diethelm K. The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type. Springer; 2010.
21. Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. In: North-Holland Mathematics Studies. Elsevier Science B.V.; 2006. Volume 204.
22. Miller KS, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley-Interscience; 1993.
23. Podlubny I. Fractional Differential Equations. Academic Press; 1999.
24. Eloe P, Lyons JW, Neugebauer JT. An ordering on Green's functions for a family of two-point boundary value problems for fractional differential equations. Communications in Applied Analysis. 2015; 19: 453–462.