Mathematics and Systems Science 2025, 3(2), 3273.
https://doi.org/10.54517/mss3273

Article

Generalization of a variant of k-plane trees

Fidel Ochieng Oduol[1], **Isaac Owino Okoth**[2,*], **Fredrick Oluoch Nyamwala**[1]

[1] Department of Mathematics, Physics and Computing, Moi University, P.O. Box 3900-30100 Eldoret, Kenya
[2] Department of Pure and Applied Mathematics, Maseno University, P.O. Box 333-40105 Maseno, Kenya
***Corresponding author:** Isaac Owino Okoth, ookoth@maseno.ac.ke

CITATION

Oduol FO, Okoth IO, Nyamwala FO. Generalization of a variant of k-plane trees. Mathematics and Systems Science. 2025; 3(2): 3273. https://doi.org/10.54517/mss3273

ARTICLE INFO

Received: 1 February 2025
Accepted: 20 May 2025
Available online: 12 June 2025

Abstract: Enumeration of plane trees and noncrossing trees was recently unified by considering d-dimensional plane trees in which ordinary plane trees are 1-dimensional plane trees and noncrossing trees are 2-dimensional plane trees. Also, recently variants of k-plane trees and k-noncrossing trees were introduced and enumerated according to number of nodes, root degree, label of the eldest or youngest child of the root, length of the leftmost path and number of forests with a given number of components. In this paper, we have generalized a variant of k-plane trees and k-noncrossing trees to a d-dimensional version and obtained closed formulas for the trees based on the aforementioned parameters. We have used symbolic method to find the generating functions, obtained the right substitution to solve the generating functions and applied Lagrange-Bürmann inversion to obtain the formulas. The results of this paper unify known results in the counting of k-plane trees and k-noncrossing trees.

Keywords: k_1-plane tree; d-dimensional k_1-plane tree; root degree; eldest child; youngest child; leftmost path; forest
MSC Classifcation: 05C05; 05C30

1. Introduction

Trees, which are acyclic connected simple graphs, are one of the most studied discrete structures both in combinatorial mathematics and computer science. These trees include labelled trees, plane trees, m-ary trees and noncrossing trees among other classes of trees. Of much interest in this paper are plane trees and noncrossing trees. A *plane tree* (also called *ordered tree*) is a tree in which one of its nodes is identified as the root and all its subtrees posses an ordering [1]. On the other hand, a *noncrossing tree* is a plane tree with nodes on the circumference of a circle so that edges do not intersect inside the circle [2].

Consider a plane tree P. A node e in P resides on *level* $\ell \geq 0$ if there are ℓ edges on the path from the root to e. This implies that the root is on level 0. A node i is a *child* (resp. *parent*) of node j if i is adjacent to j and j resides on level ℓ whenever i resides on level $\ell + 1$ (resp. $\ell - 1$). A non-root node with no child is a *leaf* and an *internal node* has at least one child. The *degree* of a node in a tree is the number of nodes that are adjacent to it. A *degree sequence* of a tree is a sequential arrangement of degrees of all nodes in the tree. A collection of trees is a *forest*. Plane trees have been enumerated according to a number of parameters: number of nodes [1], root degree, leaves [3], level of a node [3,4], degree sequence [5], forests [6] et cetera. Plane trees are one of the many structures counted by well studied Catalan numbers [7, A000108] and have been generalized by assigning labels to the nodes such that certain coherence conditions are satisfied [8,9]. In [8], Gu and Prodinger generalized plane trees by introducing 2-*plane trees*. These are plane trees in which the nodes are labelled 1 or 2 such that the sum of labels of adjacent vertices does not exceed 3. They found a counting formula for these trees given the number of nodes and label of the root. The work was extended in [9] by Gu et al. The authors introduced and studied k-*plane trees* which are plane trees with nodes labelled with integers in the set $\{1, 2, \ldots, k\}$ such that the sum of labels of adjacent nodes does not exceed $k + 1$. The aforesaid authors enumerated k-plane trees by number of nodes and label of the root. It is worth noting that 1-plane trees are plane trees. In 2024, Oduol et al. [10] introduced the set of k_1-*plane trees* and enumerated them by number of nodes, root degree, label of the eldest child of the root, label of the youngest child of the root, length of the leftmost path and number of forests. These are k-plane trees in which all nodes labelled 1 must be on the left of all others. In this work, we generalize k_1-plane trees and enumerate them according to the stated parameters.

Consider a noncrossing tree T. A *degree* of a node r of T is the number of edges incident to it. A node of degree 1 is called an *endpoint* and a non-root endpoint is a *leaf*. An arrangement of all degrees of a tree is its *degree sequence*. Let (u, v) be an edge in a noncrossing tree such that there are ℓ and $\ell + 1$ edges on the path from the root to u and v respectively. If $u < v$ (resp. $u > v$) then (u, v) is an *ascent* (resp. a *descent*). In 2002, Panholzer and Prodinger introduced (l, r)-representation of a noncrossing tree in [11]. This is a representation of a noncrossing tree as a plane tree where each non-root node in the noncrossing tree is labelled r (resp. l) in the plane tree if it is an ascent (resp. a descent) node and the root of the plane tree corresponds to node 1 in the noncrossing tree. This is depicted in **Figure 1**.

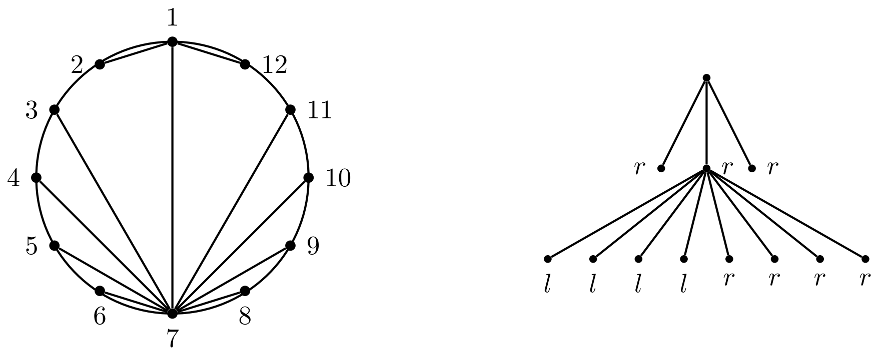

Figure 1. The (l, r)-representation of a noncrossing tree.

Since their introduction in 1998, noncrossing trees have been enumerated by number of parameters such as number of nodes [2], leaves, degree sequence, forests [12], descents [13], endpoints, maximum degree [14].

As noted for plane trees, noncrossing trees have also been generalized by giving labels to the nodes of the trees such that a coherence condition is satisfied [15,16]. In 2009, Yan and Liu [15] introduced and enumerated 2-*noncrossing trees*. These are noncrossing trees in which the nodes are labelled 1 or 2 such that if (i, j) is an ascent on the path from the root (node 1), then $i + j \leq 3$. The authors enumerated these trees by number of nodes and label of the root. A year later, Pang and Lv [16] generalized the result of Yan and Liu by introducing and studying the set of k-noncrossing trees. A k*-noncrossing tree* is a noncrossing tree in which the nodes are given integer labels in the set $\{1, \ldots, k\}$ satisfying the property that if (u, v) is an ascent on the path from the root then the sum of u and v is no more than $k + 1$. The authors obtained the number of k-noncrossing trees in which the root label and the number of nodes are given. It is also worth mentioning that 1-noncrossing trees are just ordinary noncrossing trees. In 2024, Oduol et al. [17] introduced the set of k_1-noncrossing trees. A k_1*-noncrossing tree* is a k-noncrossing tree in which in its (l, r)-representation, all ascents and descents labelled 1 are on the left of all other ascents and descents. Therein [17], the authors enumerated k_1-noncrossing trees by number of nodes, root degree, label of the eldest child of the root, label of the youngest child of the root, length of the leftmost path and forests with a given number of components. In this work, we generalize k_1-noncrossing trees and enumerate them according to the listed statistics.

The concept of butterflies introduced by Flajolet and Noy [12] has been a revelation in the enumeration of noncrossing trees. Formally, a *butterfly* comprises two ordered noncrossing trees that share a root. This means that for each node, there is a right wing and a left wing of a butterfly and each wing is a noncrossing tree. Okoth and Kasyoki [18] considered noncrossing trees having butterflies with d wings instead of two wings. There is only one right wing and the remaining $d - 1$ wings are all left wings. They called such a tree a d*-dimensional plane tree*. We note that the wings of the d-dimensional plane tree are labelled such that the left wing is the first wing and the rightmost wing is the last wing. For a convention, a wing of a butterfly rooted at the root is a right wing. **Figure 2** shows a 3-dimensional plane tree with 12 nodes.

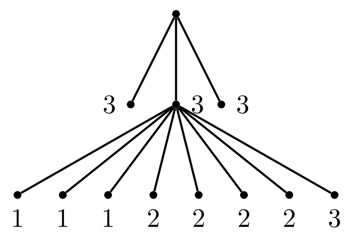

Figure 2. A 3-dimensional plane tree on 12 nodes where the labels are the wings in which a given node belongs.

It is important to note that a 1-dimensional plane tree is just a plane tree and it has no left wing. Moreover, a noncrossing tree is a 2-dimensional plane tree. In [19], Nyariaro et al. generalized k-plane trees and k-noncrossing trees by introducing the set of d-dimensional k-plane trees.

Definition 1. *A* d-dimensional k-plane tree *is a noncrossing tree (in its* (l, r)*-representation) with the property that butterflies rooted at all internal nodes have* d *wings such that if* (u, v) *is an ascent in the right wing then* $u + v \leq k + 1$ [19]. *Here, all children of the root are in the right wing.*

The authors of [19] enumerated the set of d-dimensional k-plane trees by number of nodes, root degree, label of the eldest child of the root, label of the youngest child of the root, length of the leftmost path and forest with a given number of components. In **Figure 3**, we have a 3-dimensional 4-plane tree on 12 nodes with root labelled 2.

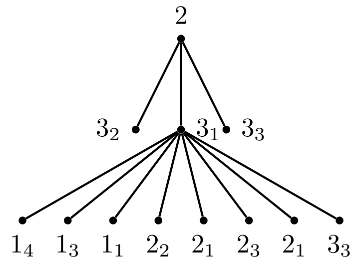

Figure 3. A 3-dimensional 4-plane tree on 12 nodes with root labelled 2 where i_j means that the node belongs to the i-th wing and is labelled j for $j \in \{1, 2, 3, 4\}$.

We remark that a 1-dimensional k-plane tree and a 2-dimensional k-plane tree are a k-plane tree and a k-noncrossing tree respectively. In the following definition, we introduce the main object of our study.

Definition 2. *A* d-dimensional k_1-plane tree *is a d-dimensional k-plane tree in which all the vertices labelled* 1 *must be on the left of all others.*

We note that a 1-dimensional k_1-plane tree is a k_1-plane tree and a 2-dimensional k_1-plane tree is a k_1-noncrossing tree. In **Figure 4**, we get a 3-dimensional 3_1-plane tree on 12 nodes with root labelled 2.

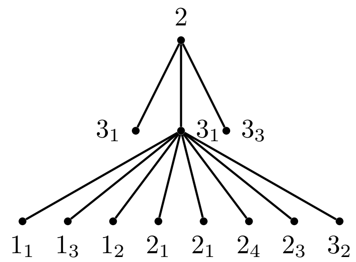

Figure 4. A 3-dimensional 4_1-plane tree on 12 nodes with root labelled 2 where i_j means that the node belongs to the i-th wing and is labelled j for $j \in \{1, 2, 3, 4\}$.

Let $N(z)$ and $B(z)$ be respectively the generating functions for d-dimensional plane trees and butterflies where z marks a node. Each tree is a node together with a sequence of butterflies. This translates to

$$N(z) = \frac{z}{1 - B(z)} \tag{1}$$

Since each butterfly consists of d noncrossing trees glued together, then

$$B(z) = \frac{N(z)^d}{z^{d-1}} \tag{2}$$

We note that $N(z)^d$ is divided by z^{d-1} since when we glue together d noncrossing trees (at a node) to form a butterfly, the number of nodes reduces by $d-1$. The division is done to avoid over counting of nodes. Substituting (2) in Equation (1), we obtain

$$N(z) = \frac{z}{1 - \dfrac{N(z)^d}{z^{d-1}}} \tag{3}$$

which is the generating function for d-dimensional plane trees. Let $\frac{N(z)}{(\sqrt[d]{z})^{d-1}} = M(z)$ so that (3) reduces to

$$M(z) = \frac{(\sqrt[d]{z})^{d-1}}{1 - M(z)^d}.$$

The following theorem enables the extraction of the coefficient of z^n in the generating function $M(z)$.

Theorem 1 (Lagrange-Bürmann inversion, [6,20])**.** *Let $M(z)$ be a generating function satisfying $M(z) = z\psi(M(z))$, where $\psi(0) \neq 0$. Then, $n[z^n]F(M(z)) = [m^{n-1}](F'(m)\psi(m)^n)$ where F is an arbitrary analytic function.*

In this paper, we use generating functions (and Lagrange-Bürmann inversion) to enumerate d-dimensional k_1-plane trees according to the number of nodes in Section 2 and root degree in Section 3. In Sections 4 and 5, we enumerate these trees by label of the eldest or youngest child of the root and length of the leftmost path respectively. The work is extended to obtain enumerative formulas for these trees by number of forests in Section 6. Section 7 gives a brief summary of the results and an array of ideas on how the work could be extended.

2. Number of nodes

In the sequel, we get our first result.

Theorem 2. *There are*

$$\frac{1}{d(n-1)+1} \sum_{a=0}^{n-2} \frac{(k-i)(d(n-1)+1) + da(i-1)}{n-a-1} \binom{d(n-1)+a}{a} \binom{(k-1)(d(n-1)+1)-i}{n-a-2} \tag{4}$$

d-dimensional k_1-plane trees with n nodes whose root is labelled i.

Proof. Consider the set of d-dimensional k_1-plane trees. If one of the endpoints of an ascent edge (in the first wing) in d-dimensional k_1-plane tree is labelled i then the other endpoint must have a label no more than $k - i + 1$. Let $N_i(z) = N_i$ be the generating function for d-dimensional k_1-plane trees with roots labelled by i where z marks a node. Let $B_i(z)$ be the generating function of a butterfly rooted at a node labelled i where z marks a node. Since only the trees in the d-th wing satisfy the ascent rule, then the trees in the first $d-1$ wings can be considered to have the root labelled 1. So $B_i(z) = \frac{N_1^{d-1}N_i}{z^{d-1}}$. The division by z^{d-1} is to ensure that nodes are not over counted. For each d-dimensional k_1-plane tree with root labelled i, there is a sequence of subtrees with roots labelled 1 that appear on the left, followed by a sequence of subtrees with roots labelled j for $j = 2, 3, \ldots, k-i+1$ that are attached to the root (marked by z). By the product rule, the generating function for N_i is thus given by

$$N_i(z) = z \cdot \frac{1}{1 - \frac{N_1^d}{z^{d-1}}} \cdot \frac{1}{1 - \frac{N_1^{d-1}}{z^{d-1}}(N_2 + \cdots + N_{k-i+1})} \tag{5}$$

To obtain the suitable substitution to solve the system of the functional equations

(5), let $N_i(z) = \dfrac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}$ and $z = \dfrac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}(1-w)}{(1+w)^{k-1}}$. From the system (5), we have

$$N_i(z) = z \cdot \frac{1}{1 - \frac{((\sqrt[d]{z})^{d-1}\sqrt[d]{w})^d}{z^{d-1}}} \cdot \frac{1}{1 - \frac{((\sqrt[d]{z})^{d-1}\sqrt[d]{w})^{d-1}}{z^{d-1}}\left(\frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{1+w} + \cdots + \frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{k-i}}\right)},$$

which is the same as

$$N_i(z) = z \cdot \frac{1}{1-w} \cdot \frac{1}{1 - \frac{w}{1+w}\left(1 + \frac{1}{1+w} + \cdots + \frac{1}{(1+w)^{k-i-1}}\right)}.$$

Summing the geometric series, we get

$$\begin{aligned} N_i(z) &= z \cdot \frac{1}{1-w} \cdot \frac{1}{1-(1-(1+w)^{i-k})} = z \cdot \frac{1}{1-w} \cdot \frac{1}{(1+w)^{i-k}} \\ &= \frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}(1-w)}{(1+w)^{k-1}} \cdot \frac{1}{1-w} \cdot \frac{1}{(1+w)^{i-k}} = \frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}. \end{aligned}$$

Since the substitutions $N_i(z) = \dfrac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}$ and $z = \dfrac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}(1-w)}{(1+w)^{k-1}}$ satisfy (5) and the latter is independent of i then these are the rights substitutions to solve the system of functional equations (5). This implies that $w = z(1-w)^{-d}(1+w)^{d(k-1)}$. We use Lagrange-Bürmann inversion to extract the coefficient of z^n in w.

We have,

$$\begin{aligned} [z^n]N_i =& [z^n]\frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}} = [z^{n-(d-1)/d}]\sqrt[d]{w}(1+w)^{-(i-1)} \\ =& \frac{1}{n-\frac{d-1}{d}}[w^{n-1-(d-1)/d}]\left(\frac{1}{d(\sqrt[d]{w})^{d-1}}(1+w)^{-(i-1)} - (i-1)\sqrt[d]{w}(1+w)^{-i}\right) \\ & (1-w)^{-d(n-(d-1)/d)}(1+w)^{d(k-1)(n-(d-1)/d)} \\ =& \frac{1}{d(n-1)+1}[w^{n-1}]\left(1-(d(i-1)-1)w\right)(1-w)^{-(d(n-1)+1)}(1+w)^{(k-1)(d(n-1)+1)-i}. \end{aligned}$$

By binomial theorem, we obtain

$$\begin{aligned} [z^n]N_i =& \frac{1}{d(n-1)+1}[w^{n-1}]\left(1-(d(i-1)-1)w\right)\sum_{a,b\geq 0}\binom{d(n-1)+a}{a}\binom{(k-1)(d(n-1)+1)-i}{b}w^{a+b} \\ =& \frac{1}{d(n-1)+1}\sum_{a\geq 0}\binom{d(n-1)+a}{a}\left[\binom{(k-1)(d(n-1)+1)-i}{n-a-1}\right. \\ & \left. -(d(i-1)-1)\binom{(k-1)(d(n-1)+1)-i}{n-a-2}\right] \\ =& \frac{1}{d(n-1)+1}\sum_{a=0}^{n-2}\frac{(k-i)(d(n-1)+1)+da(i-1)}{n-a-1}\binom{d(n-1)+a}{a}\binom{(k-1)(d(n-1)+1)-i}{n-a-2}. \end{aligned}$$

□

Setting $i = 1$ in Equation (4), we find that there are

$$\frac{1}{d(n-1)+1}\sum_{a=0}^{n-1}\binom{d(n-1)+a}{a}\binom{(k-1)(d(n-1)+1)}{n-a-1} \tag{6}$$

d-dimensional k_1-plane trees with n nodes whose root is labelled 1. By setting $d = 1$ and $d = 2$ in Equation (6), we rediscover the formulas for the number of k_1-plane trees and k_1-noncrossing trees on n nodes whose root is labelled 1 which were obtained in [10] and [17] respectively.

Also setting $i = k$ in Equation (4), we get the formula for d-dimensional k_1-plane trees with n nodes such that the root is labelled k as

$$\frac{1}{d(n-1)+1}\sum_{a=0}^{n-1}\frac{a}{n-1}\binom{d(n-1)+a}{a}\binom{d(k-1)(n-1)}{n-a-1} \tag{7}$$

Further setting $d = 1$ in Equation (7), we get

$$\frac{1}{n}\sum_{a=0}^{n-1}\frac{a}{n-1}\binom{n+a-1}{a}\binom{(k-1)(n-1)}{n-a-1} \tag{8}$$

as the formula for the number of k_1-plane trees with n nodes such that the root is labelled k. Equation (8) was obtained by Oduol et al. in [10]. Moreover, setting $d = 2$ in Equation (7), we have

$$\frac{1}{2n-1}\sum_{a=0}^{n-1}\frac{a}{n-1}\binom{2n+a-2}{a}\binom{(k-1)(2n-2)}{n-a-1} \tag{9}$$

as the formula for the number of k_1-noncrossing trees with n nodes such that the root is labelled k. Equation (9) was derived by Oduol et al. in [17].

Now, we get the formula

$$\frac{1}{d(n-1)+1}\binom{(d+1)(n-1)}{n-1} \tag{10}$$

for the number of d-dimensional plane trees on n nodes upon letting $k = 1$ in Equation (7). This formula was obtained by Okoth and Kasyoki in [18] with the cases $d = 1$ and $d = 2$ obtained earlier in [3] and [2] respectively.

3. Root degree

In this section, we enumerate the set of d-dimensional k_1-plane trees according to root degree.

Theorem 3. *There are*

$$\frac{r}{n-1}\sum_{a=0}^{n-r-1}\frac{(d(k-1)-i+1)(n-1)+a(i-1)}{d(k-1)(n-1)-(i-1)r}\binom{d(n-1)+a-1}{a}\binom{d(k-1)(n-1)-(i-1)r}{n-r-a-1} \tag{11}$$

d-dimensional k_1-plane trees on n nodes with root labelled j of degree r such that all children of the root are labelled i.

Proof. Let $N_i(z) = N_i$ be the generating function for d-dimensional k_1-plane trees with roots labelled by i where z marks a node. Let $B_i(z)$ be the generating function of a butterfly rooted at a node labelled i where z marks a node. Since only the trees in the d-th wing satisfy the ascent rule, then the trees in the first $d-1$ wings can be considered to have the root labelled 1. So $B_i(z) = \frac{N_1^{d-1}N_i}{z^{d-1}}$. The division by z^{d-1} is to ensure that we avoid over counting nodes. Since the root is marked z then the generating function for d-dimensional k_1-plane trees with root degree r such that all the children of the root are labelled i is $z\left(\frac{N_1^{d-1}N_i}{z^{d-1}}\right)^r$. As in the previous section, we use the substitution $N_i(z) = \dfrac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}$ where $w = z(1-w)^{-d}(1+w)^{d(k-1)}$.

We now extract the coefficient of z^n in $z\left(\frac{N_1^{d-1}N_i}{z^{d-1}}\right)^r$:

$$
\begin{aligned}
[z^n]z\left(\frac{N_1^{d-1}N_i}{z^{d-1}}\right)^r &= [z^{n-1}]\left(\frac{N_1^{d-1}N_i}{z^{d-1}}\right)^r = [z^{n-1}]\left(\frac{((\sqrt[d]{z})^{d-1}\sqrt[d]{w})^{d-1}}{z^{d-1}}\right)^r\left(\frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}\right)^r \\
&= [z^{n-1}]w^r(1+w)^{-(i-1)r}.
\end{aligned}
$$

By Lagrange-Bürmann inversion, we obtain

$$
\begin{aligned}
[z^n]z\left(\frac{N_1^{d-1}N_i}{z^{d-1}}\right)^r &= \frac{1}{n-1}[w^{n-2}]\left(rw^{r-1}(1+w)^{-(i-1)r} - (i-1)rw^r(1+w)^{-(i-1)r-1}\right) \\
&\quad (1-w)^{-d(n-1)}(1+w)^{d(k-1)(n-1)} \\
&= \frac{r}{n-1}[w^{n-r-1}]\left(1-(i-2)w\right)(1-w)^{-d(n-1)}(1+w)^{d(k-1)(n-1)-(i-1)r-1}.
\end{aligned}
$$

Binomial theorem gives

$$
\begin{aligned}
[z^n]z\left(\frac{N_1^{d-1}N_i}{z^{d-1}}\right)^r &= \frac{r}{n-1}[w^{n-r-1}]\left(1-(i-2)w\right) \\
&\quad \sum_{a,b\geq 0}\binom{d(n-1)+a-1}{a}\binom{d(k-1)(n-1)-(i-1)r-1}{b}w^{a+b} \\
&= \frac{r}{n-1}\sum_{a\geq 0}\binom{d(n-1)+a-1}{a} \\
&\quad \left[\binom{d(k-1)(n-1)-(i-1)r-1}{n-r-a-1} - (i-2)\binom{d(k-1)(n-1)-(i-1)r-1}{n-r-a-2}\right] \\
&= \frac{r}{n-1}\sum_{a=0}^{n-r-1}\frac{(d(k-1)-i+1)(n-1)+a(i-1)}{d(k-1)(n-1)-(i-1)r}\binom{d(n-1)+a-1}{a} \\
&\quad \binom{d(k-1)(n-1)-(i-1)r}{n-r-a-1}.
\end{aligned}
$$

□

Setting $d = 1$ in Equation (11), we find the following corollary that was obtained by Oduol et al. in [10].

Corollary 1. *There are*

$$\frac{r}{n-1}\sum_{a=0}^{n-r-1}\frac{(k-i)(n-1)+a(i-1)}{(k-1)(n-1)-(i-1)r}\binom{(n-1)+a-1}{a}\binom{(k-1)(n-1)-(i-1)r}{n-r-a-1},$$

k_1-plane trees on n nodes with root labelled j such that the root has r children, all of which are labelled i.

Also, setting $d=2$ in Equation (11), we rediscover the formula for the number of k_1-noncrossing trees with a given root degree. The result was initially obtained by Oduol et al. in [17].

Corollary 2. *The number of k_1-noncrossing trees on n nodes with root labelled j such that the root degree is r and all the children of the root are labelled i is given by*

$$\frac{r}{n-1}\sum_{a=0}^{n-r-1}\frac{(2k-i-1)(n-1)+a(i-1)}{2(k-1)(n-1)-(i-1)r}\binom{2(n-1)+a-1}{a}\binom{2(k-1)(n-1)-(i-1)r}{n-r-a-1}.$$

Setting $i=k$ in Equation (11), we find that there are

$$\frac{r}{n-1}\sum_{a=0}^{n-r-1}\frac{(d-1)(n-1)+a}{d(n-1)-r}\binom{d(n-1)+a-1}{a}\binom{(k-1)(d(n-1)-r)}{n-r-a-1} \tag{12}$$

d-dimensional k_1-plane trees on n nodes with root labelled 1 such that the root has degree r and all the children of the root are labelled k. Now, if we set $k=1$ in Equation (12), then we have that there are

$$\frac{r}{n-1}\binom{d(n-1)+(n-r-1)-1}{n-r-1}=\frac{r}{n-1}\binom{(d+1)(n-1)-r-1}{n-r-1} \tag{13}$$

d-dimensional plane trees on n nodes with a root of degree r. Equation (13) was derived by Okoth and Kasyoki in [18].

We get the following result upon setting $i=1$ in Equation (11):

Corollary 3. *There are*

$$\frac{r}{n-1}\sum_{a=0}^{n-r-1}\binom{d(n-1)+a-1}{a}\binom{d(k-1)(n-1)}{n-r-a-1} \tag{14}$$

d-dimensional k_1-plane trees on n nodes with root labelled k such that the root is of degree r.

We also arrive at Equation (13) by setting $k=1$ in Equation (14). If $d=1$ and $d=2$ in Equation (14) we obtain the formulas

$$\frac{r}{n-1}\sum_{a=0}^{n-r-1}\binom{n+a-2}{a}\binom{(k-1)(n-1)}{n-r-a-1},$$

and

$$\frac{r}{n-1}\sum_{a=0}^{n-r-1}\binom{2n+a-3}{a}\binom{2(k-1)(n-1)}{n-r-a-1},$$

for the number of k_1-plane trees and k_1-noncrossing trees on n nodes with root labelled k and of degree r which were initially obtained in [10] and [17] respectively.

We find a more general result in the following theorem.

Theorem 4. *There are*

$$\frac{1}{n-1}\sum_{a=0}^{n-r-1}\frac{(rd(k-1)-t)(n-1)+at}{d(k-1)(n-1)-t}\binom{d(k-1)(n-1)-t}{n-a-r-1}\binom{d(n-1)+a-1}{a}\binom{r-r_1}{r_2,r_3,\dots,r_{k-i+1}} \tag{15}$$

d-dimensional k_1-plane trees on n nodes whose root is labelled i and the root has r children, r_j of them are labelled j where $j=1,2,\dots,k-i+1$ and $t:=r_2+2r_3+\dots+(k-i)r_{k-i+1}$.

Proof. Let $N_i(z)$ be the generating function for d-dimensional k_1-plane trees rooted at a node labelled i, where z marks a node. Let $B_i(z)$ be the generating function of a butterfly rooted at a node labelled i where z marks a node. Since only the trees in the d-th wing satisfy the ascent rule, then the trees in the first $d-1$ wings can be considered to have the root labelled 1. So $B_i(z)=\frac{N_1^{d-1}N_i}{z^{d-1}}$. Since the root is marked z and there are r_i subtrees rooted at the children of the root for $i=1,2,\dots,k$, the generating function for the required d-dimensional k_1-plane trees in which the position of the subtrees is not taken into consideration is

$$z\left(\frac{N_1(z)^d}{z^{d-1}}\right)^{r_1}\left(\frac{N_1(z)^{d-1}N_2(z)}{z^{d-1}}\right)^{r_2}\cdots\left(\frac{N_1(z)^{d-1}N_{k-i+1}(z)}{z^{d-1}}\right)^{r_{k-i+1}}=z^{(1-d)r+1}N_1^{r(d-1)}N_1^{r_1}N_2^{r_2}\cdots N_{k-i+1}^{r_{k-i+1}}.$$

We extract the coefficient z^n in the generating function.

$$\begin{aligned}
[z^n]z^{(1-d)r+1}N_1^{r(d-1)}N_1^{r_1}N_2^{r_2}\cdots N_{k-i+1}^{r_{k-i+1}}&=[z^{n+(d-1)r-1}]N_1^{r(d-1)}N_1^{r_1}N_2^{r_2}\cdots N_{k-i+1}^{r_{k-i+1}}\\
&=[z^{n+(d-1)r-1}]((\sqrt[d]{z})^{d-1}\sqrt[d]{w})^{r(d-1)}((\sqrt[d]{z})^{d-1}\sqrt[d]{w})^{r_1}\cdot\left(\frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{1+w}\right)^{r_2}\cdots\left(\frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{k-i}}\right)^{r_{k-i+1}}\\
&=[z^{n+(d-1)r-1}]z^{r(d-1)}w^r\cdot\left(\frac{1}{1+w}\right)^{r_2}\cdots\left(\frac{1}{(1+w)^{k-i}}\right)^{r_{k-i+1}}\\
&=[z^{n-1}]w^r(1+w)^{-t},
\end{aligned}$$

where $w=z(1-w)^{-d}(1+w)^{d(k-1)}$ and $t:=r_2+2r_3+\dots+(k-i)r_{k-i+1}$.

By Lagrange-Bürmann inversion, we have,

$$\begin{aligned}
[z^n]z^{(1-d)r+1}N_1^{r(d-1)}N_1^{r_1}N_2^{r_2}\cdots N_{k-i+1}^{r_{k-i+1}}&=\frac{1}{n-1}[w^{n-2}](rw^{r-1}(1+w)^{-t}-tw^r(1+w)^{-t-1})\\
&(1-w)^{-d(n-1)}(1+w)^{d(k-1)(n-1)}\\
&=\frac{1}{n-1}(r[w^{n-r-1}](1+w)^{d(k-1)(n-1)-t}-t[w^{n-r-2}](1+w)^{d(k-1)(n-1)-t-1})\\
&(1-w)^{-d(n-1)}\\
&=\frac{1}{n-1}\sum_{a\geq 0}\left[r\binom{d(k-1)(n-1)-t}{n-a-r-1}-t\binom{d(k-1)(n-1)-t-1}{n-a-r-2}\right]\binom{d(n-1)+a-1}{a}\\
&=\frac{1}{n-1}\sum_{a=0}^{n-r-1}\frac{(dr(k-1)-t)(n-1)+at}{d(k-1)(n-1)-t}\binom{d(k-1)(n-1)-t}{n-a-r-1}\binom{d(n-1)+a-1}{a}.
\end{aligned}$$

Now, since all the children labelled 1 for each internal node are on the left of all others then there are

$$\binom{r-r_1}{r_2, r_3, \ldots, r_{k-i+1}},$$

ways of assigning labels to the children of the root so that there are r_j children labelled j for $j = 1, 2, \ldots, k-i+1$. The proof follows by the product rule of counting. $\square$

Equation (11) follows from Equation (15), by setting $t = r(i-1)$ and $r_j = 0$ for all $j \neq i$. If $t = 0$ in Theorem 4 then $r_1 = r$, $r_2 = r_3 = \cdots = r_{k-i+1} = 0$. Then it follows that there are

$$\frac{r}{n-1} \sum_{a=0}^{n-r-1} \binom{d(k-1)(n-1)}{n-a-r-1} \binom{d(n-1)+a-1}{a} \tag{16}$$

d-dimensional k_1-plane trees on n nodes such that the root is labelled k and is of degree r. Equation (16) was also obtained in Equation (14).

If $k = 2$ and $i = 1$ in Equation (15) then $r_1 + r_2 = r$ and $r_2 = t$. This implies that $r_2 = r - r_1$ and $t = r - r_1$. We then obtain that there are

$$\frac{1}{n-1} \sum_{a=0}^{n-r-1} \frac{((d-1)r + r_1)(n-1) + a(r-r_1)}{d(n-1) - r + r_1} \binom{d(n-1) - r + r_1}{n-a-r-1} \binom{d(n-1)+a-1}{a} \tag{17}$$

d-dimensional 2_1-plane trees on n nodes with root labelled 1 and r children of which r_1 are labelled 1. Summing over all values of r_1 and r in (17), we find the total number of d-dimensional 2_1-plane trees on n nodes with root labelled 1.

If $k = 2$ and $i = 2$ in (15) then $r_1 = r$ and $t = 0$. It means that there

$$\frac{r}{n-1} \sum_{a=0}^{n-r-1} \binom{d(n-1)}{n-a-r-1} \binom{d(n-1)+a-1}{a},$$

d-dimensional 2_1-plane trees on n nodes with root labelled 2 and r children all labelled 1.

4. Eldest or youngest child of the root

This section is dedicated to obtaining counting formulas for d-dimensional k_1-plane trees in which the label of the eldest or youngest child of the root is taken into consideration. We start with the case in which the eldest child of the root is labelled 1.

Theorem 5. *The number of d-dimensional k_1-plane trees on n nodes whose root is labelled i such that the eldest child of the root is labelled 1 is given by*

$$\frac{1}{d(n-1)+1} \sum_{a=0}^{n-2} \frac{(dk+k-i-d)(d(n-1)+1) + da(i-1)}{(k-1)(d(n-1)+1) - i + 1} \binom{d(n-1)+a}{a} \binom{(k-1)(d(n-1)+1) - i + 1}{n-a-2}. \tag{18}$$

Proof. Deletion of the edge connecting the root to its eldest child implies that the required generating function is a product of the generating function of the butterfly rooted at the eldest child of the root and that of the d-dimensional k_1-plane tree with root labelled i. The generating function of the trees in the statement of the theorem is thus given by $\dfrac{N_1(z)^d}{z^{d-1}} \cdot N_i(z)$. Again, as in Section 2, the right substitution is $N_i(z) = \dfrac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}$ where $w = z(1-w)^{-d}(1+w)^{d(k-1)}$. We extract the coefficient of z^n in the generating function.

$$[z^n]\frac{N_1(z)^d}{z^{d-1}} \cdot N_i(z) = [z^n]\frac{\left((\sqrt[d]{z})^{d-1}\sqrt[d]{w}\right)^d}{z^{d-1}} \cdot \frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}} = [z^{n+d-1}]z^{(d+1)(d-1)/d}w^{(d+1)/d}(1+w)^{1-i}$$
$$= [z^{(d(n-1)+1)/d}]w^{(d+1)/d}(1+w)^{1-i}.$$

Application of Lagrange Bürmann inversion gives

$$\begin{aligned}
&[z^n]\frac{N_1(z)^d}{z^{d-1}} \cdot N_i(z)\\
&= \frac{1}{(d(n-1)+1)/d}[w^{(d(n-1)+1)/d-1}]\left(\frac{d+1}{d}w^{1/d}(1+w)^{1-i} - (i-1)(1+w)^{-i}w^{(d+1)/d}\right)\\
&(1-w)^{-d((d(n-1)+1)/d)}(1+w)^{d(k-1)((d(n-1)+1)/d)}\\
&= \frac{1}{d(n-1)+1}[w^{n-2}]\left(d+1-w(d(i-2)-1)\right)(1-w)^{-(d(n-1)+1)}(1+w)^{(k-1)(d(n-1)+1)-i}\\
&= \frac{1}{d(n-1)+1}[w^{n-2}]\left(d+1-w(d(i-2)-1)\right)\\
&\sum_{a,b\geq 0}\binom{d(n-1)+a}{a}\binom{(k-1)(d(n-1)+1)-i}{b}w^{a+b}\\
=&\frac{1}{d(n-1)+1}\sum_{a=0}^{n-3}\binom{d(n-1)+a}{a}\left[(d+1)\binom{(k-1)(d(n-1)+1)-i}{n-a-2}\right.\\
&\left.-(d(i-2)-1)\binom{(k-1)(d(n-1)+1)-i}{n-a-3}\right]\\
=&\frac{1}{d(n-1)+1}\sum_{a=0}^{n-2}\left[\frac{(dk+k-i-d)(d(n-1)+1)+da(i-1)}{(k-1)(d(n-1)+1)-i+1}\binom{d(n-1)+a}{a}\right.\\
&\left.\binom{(k-1)(d(n-1)+1)-i+1}{n-a-2}\right].
\end{aligned}$$

□

Setting $i = 1$ in Theorem 18, we get the following corollary.

Corollary 4. *The number of d-dimensional k_1-plane trees on n nodes whose root is labelled 1 such that the eldest child of the root is also labelled 1 is given by*

$$\frac{d+1}{d(n-1)+1}\sum_{a=0}^{n-2}\binom{d(n-1)+a}{a}\binom{(k-1)(d(n-1)+1)}{n-a-2} \tag{19}$$

If we set $d = 1$ in Equation (18), we obtain that there are

$$\frac{1}{n}\sum_{a=0}^{n-2}\frac{(2k-i-1)n+a(i-1)}{(k-1)n-i+1}\binom{n+a-1}{a}\binom{(k-1)n-i+1}{n-a-2},$$

k_1-plane trees on n nodes in which the root is labelled i and the eldest child of the root is labelled 1. This result was also obtained by the authors of [10].

Also, setting $d=2$ in Equation (18), we obtain the following corollary.

Corollary 5. *There are*

$$\frac{1}{2n-1}\sum_{a=0}^{n-2}\frac{(3k-i-2)(2n-1)+2a(i-1)}{(k-1)(2n-1)-i+1}\binom{2(n-1)+a}{a}\binom{(k-1)(2n-1)-i+1}{n-a-2},$$

k_1-noncrossing trees on n nodes in which the root is labelled i and the eldest child of the root is labelled 1 (see [17]).

We now enumerate d-dimensional k_1-plane trees according to the label of the youngest child of the root.

Theorem 6. *The number of d-dimensional k_1-plane trees on n nodes whose root is labelled i such that the youngest child of the root is labelled $j \neq 1$ is given by*

$$\begin{aligned}\frac{1}{d(n-1)+1}\sum_{a=0}^{n-2}\Bigg[&\frac{(dk+k-i-j-d+1)(d(n-1)+1)+da(i+j-2)}{(k-1)(d(n-1)+1)-i-j+2}\binom{d(n-1)+a}{a}\\&\binom{(k-1)(d(n-1)+1)-i-j+2}{n-a-2}\Bigg]\end{aligned} \tag{20}$$

Proof. Deletion of the edge connecting the root to its youngest child implies that the required generating function is a product of the generating function of the butterfly rooted at the youngest child of the root and that of the d-dimensional k_1-plane tree with root labelled i. The generating function of the trees in the statement of the theorem is thus given by $\dfrac{N_1(z)^{d-1}N_j(z)}{z^{d-1}}\cdot N_i(z)$. As verified in Section 2, the right substitution to solve the generating function is $N_i(z)=\dfrac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}$ where $w=z(1-w)^{-d}(1+w)^{d(k-1)}$. We now proceed to extract the coefficient of z^n in the generating function.

$$\begin{aligned}[z^n]\frac{N_1(z)^{d-1}N_j(z)}{z^{d-1}}N_i(z)&=[z^n]\frac{\left((\sqrt[d]{z})^{d-1}\sqrt[d]{w}\right)^d}{z^{d-1}(1+w)^{j-1}}\cdot\frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}\\&=[z^{n+d-1}]z^{(d+1)(d-1)/d}w^{(d+1)/d}(1+w)^{2-i-j}\\&=[z^{(d(n-1)+1)/d}]w^{(d+1)/d}(1+w)^{2-i-j}.\end{aligned}$$

By Lagrange-Bürmann inversion, we have

$$\begin{aligned}&[z^n]\frac{N_1(z)^{d-1}N_j(z)}{z^{d-1}}N_i(z)\\&\quad=\frac{1}{(d(n-1)+1)/d}[w^{(d(n-1)+1)/d-1}]\left(\frac{d+1}{d}w^{1/d}(1+w)^{2-i-j}-(i+j-2)(1+w)^{1-i-j}w^{(d+1)/d}\right)\\&\quad(1-w)^{-d((d(n-1)+1)/d)}(1+w)^{d(k-1)((d(n-1)+1)/d)}\\&\quad=\frac{1}{d(n-1)+1}[w^{n-2}]\left(d+1-w(d(i+j-3)-1)\right)(1-w)^{-(d(n-1)+1)}(1+w)^{(k-1)(d(n-1)+1)-i-j+1}.\end{aligned}$$

Making use of binomial theorem, we get

$$
\begin{aligned}
&[z^n]\frac{N_1(z)^{d-1}N_j(z)}{z^{d-1}}N_i(z)\\
=&\frac{1}{d(n-1)+1}[w^{n-2}]\left(d+1-w(d(i+j-3)-1)\right)\\
&\sum_{a,b\geq 0}\binom{d(n-1)+a}{a}\binom{(k-1)(d(n-1)+1)-i-j+1}{b}w^{a+b}\\
=&\frac{1}{d(n-1)+1}\sum_{a=0}^{n-3}\binom{d(n-1)+a}{a}\left[(d+1)\binom{(k-1)(d(n-1)+1)-i-j+1}{n-a-2}\right.\\
&\left.-(d(i+j-3)-1)\binom{(k-1)(d(n-1)+1)-i-j+1}{n-a-3}\right]\\
=&\frac{1}{d(n-1)+1}\sum_{a=0}^{n-2}\left[\frac{(dk+k-i-j-d+1)(d(n-1)+1)+da(i+j-2)}{(k-1)(d(n-1)+1)-i-j+2}\binom{d(n-1)+a}{a}\right.\\
&\left.\binom{(k-1)(d(n-1)+1)-i-j+2}{n-a-2}\right].
\end{aligned}
$$

□

The following result follows by letting $d=1$ in Equation (20).

Corollary 6. *There are*

$$\frac{1}{n}\sum_{a=0}^{n-2}\frac{(2k-i-j)n+a(i+j-2)}{(k-1)n-i-j+2}\binom{n+a-1}{a}\binom{(k-1)n-i-j+2}{n-a-2},$$

k_1-plane trees on n nodes such that the root is labelled i and the youngest child of the root is labelled $j\neq 1$, [10].

Upon setting $d=2$ in Equation (20), we obtain the following result.

Corollary 7. *The number of k_1-noncrossing trees on n nodes such that the root is labelled i and the youngest child of the root is labelled $j\neq 1$ is given by (see* [17]*)*

$$\frac{1}{2n-1}\sum_{a=0}^{n-2}\frac{(3k-i-j-1)(2n-1)+2a(i+j-2)}{(k-1)(2n-1)-i-j+2}\binom{2n+a-2}{a}\binom{(k-1)(2n-1)-i-j+2}{n-a-2}.$$

5. Leftmost path

We now switch our attention to enumeration of d-dimensional k_1-plane trees according to the length of the leftmost path.

Theorem 7. *The number of d-dimensional k_1-plane trees on n nodes whose root is labelled i such that there is a leftmost path of length $\ell\geq 1$ and all nodes on the path except the root are labelled 1 is given by*

$$\frac{1}{d(n-1)+1}\sum_{a=0}^{n-\ell-1}\frac{(d\ell k+k-i-d\ell)(d(n-1)+1)+da(i-1)}{(k-1)(d(n-1)+1)-i+1}\binom{d(n-1)+a}{a}\binom{(k-1)(d(n-1)+1)-i+1}{n-a-\ell-1}\tag{21}$$

Proof. Deletion of the ℓ edges on the leftmost path implies that the desired generating function is a product of the generating function of the ℓ butterflies rooted at the non-root vertices on the path and the generating function of the d-dimensional k_1-plane tree with root labelled i. The generating function for the trees in which there is a leftmost path of length ℓ is thus $\left(\frac{N_1(z)^d}{z^{d-1}}\right)^{\ell} N_i(z)$. From Section 2, the right substitution to solve the generating function is $N_i(z) = \frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}$ where $w = z(1-w)^{-d}(1+w)^{d(k-1)}$. So,

$$
\begin{aligned}
[z^n]\left(\frac{N_1(z)^d}{z^{d-1}}\right)^{\ell} \cdot N_i(z) &= [z^n]\left(\frac{\left((\sqrt[d]{z})^{d-1}\sqrt[d]{w}\right)^d}{z^{d-1}}\right)^{\ell} \cdot \frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}} \\
&= [z^{n+\ell(d-1)}]z^{(d\ell+1)(d-1)/d}w^{(d\ell+1)/d}(1+w)^{1-i} \\
&= [z^{(d(n-1)+1)/d}]w^{(d\ell+1)/d}(1+w)^{1-i}.
\end{aligned}
$$

We apply Lagrange Bürmann inversion to get

$$
\begin{aligned}
&[z^n]\left(\frac{N_1(z)^d}{z^{d-1}}\right)^{\ell} \cdot N_i(z) \\
&= \frac{1}{(d(n-1)+1)/d}[w^{n-\ell-1}]\left(\frac{d\ell+1}{d}w^{\ell-1+1/d}(1+w)^{1-i} - (i-1)(1+w)^{-i}w^{(d\ell+1)/d}\right) \\
&\quad (1-w)^{-d((d(n-1)+1)/d)}(1+w)^{d(k-1)((d(n-1)+1)/d)} \\
&= \frac{1}{d(n-1)+1}[w^{n-\ell-1}]\left(d\ell+1-w(d(i-\ell-1)-1)\right)(1-w)^{-(d(n-1)+1)}(1+w)^{(k-1)(d(n-1)+1)-i}.
\end{aligned}
$$

By binomial theorem, we have

$$
\begin{aligned}
[z^n]\left(\frac{N_1(z)^d}{z^{d-1}}\right)^{\ell} \cdot N_i(z) &= \frac{1}{d(n-1)+1}[w^{n-\ell-1}]\left(d\ell+1-w(d(i-\ell-1)-1)\right) \\
&\quad \sum_{a,b\geq 0}\binom{d(n-1)+a}{a}\binom{(k-1)(d(n-1)+1)-i}{b}w^{a+b} \\
&= \frac{1}{d(n-1)+1}\sum_{a=0}^{n-\ell-2}\binom{d(n-1)+a}{a}\left[(d\ell+1)\binom{(k-1)(d(n-1)+1)-i}{n-a-\ell-1}\right. \\
&\quad \left.-(d(i-\ell-1)-1)\binom{(k-1)(d(n-1)+1)-i}{n-a-\ell-2}\right] \\
&= \frac{1}{d(n-1)+1}\sum_{a=0}^{n-\ell-1}\left[\frac{(d\ell k+k-i-d\ell)(d(n-1)+1)+da(i-1)}{(k-1)(d(n-1)+1)-i+1}\binom{d(n-1)+a}{a}\right. \\
&\quad \left.\binom{(k-1)(d(n-1)+1)-i+1}{n-a-\ell-1}\right].
\end{aligned}
$$

□

Setting $d=1$ in Equation (21), we get the following corollary.

Corollary 8. *The number of k_1-plane trees on n nodes with root labelled i such that there is a leftmost path of length $\ell \geq 0$ and all the other nodes on the path are labelled 1 is given by (see* [10]*)*

$$
\frac{1}{n}\sum_{a=0}^{n-\ell-1}\frac{(\ell k+k-i-\ell)n+a(i-1)}{(k-1)n-i+1}\binom{n+a-1}{a}\binom{(k-1)n-i+1}{n-a-\ell-1}.
$$

Also, setting $k = 1$ in Equation (21), we obtain:

Corollary 9. *The number of k_1-noncrossing trees on n nodes with root labelled i such that there is a leftmost path of length $\ell \geq 0$ and all the other nodes on the path are labelled 1 is given by (see* [17]*)*

$$\frac{1}{2n-1}\sum_{a=0}^{n-\ell-1}\frac{(2\ell k+k-i-2\ell)(2n-1)+2a(i-1)}{(k-1)(2n-1)-i+1}\binom{2n+a-2}{a}\binom{(k-1)(2n-1)-i+1}{n-a-\ell-1}.$$

If $\ell = 1$ in Equation (21), we obtain the following result.

Corollary 10. *The number of d-dimensional k_1-plane trees on n nodes with root labelled i such that the eldest child of the root is labelled 1 is given by*

$$\frac{1}{d(n-1)+1}\sum_{a=0}^{n-2}\frac{(dk+k-i-d)(d(n-1)+1)+da(i-1)}{(k-1)(d(n-1)+1)-i+1}\binom{d(n-1)+a}{a}\binom{(k-1)(d(n-1)+1)-i+1}{n-a-2} \tag{22}$$

Equation (22) was already obtained in Equation (18). Moreover, if we let $\ell = 0$ in Equation (21), then we find that there are

$$\frac{1}{d(n-1)+1}\sum_{a=0}^{n-1}\frac{(k-i)(d(n-1)+1)+da(i-1)}{(k-1)(d(n-1)+1)-i+1}\binom{d(n-1)+a}{a}\binom{(k-1)(d(n-1)+1)-i+1}{n-a-1} \tag{23}$$

d-dimensional k_1-plane trees on n nodes such that the root is labelled i.

6. Forests

In this section, we enumerate forests of d-dimensional k_1-plane trees. The nodes of the forests are labelled by integers $1, 2, \ldots, n$. We shall refer to these forests as *labelled forests*. The nodes of the forests are labelled so as to avoid redundancies in counting unlabelled structures.

Theorem 8. *The number of labelled forests of d-dimensional k_1-plane trees on n nodes with r components such that for each component, the root is labelled i is given by*

$$\frac{rn!}{d(n-r)+r}\sum_{a=0}^{n-r}\frac{(k-i)(d(n-r)+r)+ad(i-1)}{d(k-1)(n-r)+(k-i)r}\binom{d(n-r)+r+a-1}{a}\binom{d(k-1)(n-r)+(k-i)r}{n-r-a}. \tag{24}$$

Proof. Let $N_i(z)$ be the generating function for d-dimensional k_1-plane trees rooted at a node labelled i, where z marks a node. Since each component of the forest is a d-dimensional k_1-plane tree then we extract the coefficient of z^n in N_i^r. As shown in Section 2, the right substitution to solve the generating function is $N_i(z) = \dfrac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}$ where $w = z(1-w)^{-d}(1+w)^{d(k-1)}$. We proceed as follows:

$$[z^n]N_i^r = [z^n]\left(\frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}\right)^r = [z^{(dn-(d-1)r)/d}]w^{r/d}(1+w)^{-(i-1)r}.$$

By Lagrange-Bürmann inversion, we have

$$[z^n]N_i^r = \frac{1}{(dn-(d-1)r)/d}[w^{(dn-(d-1)r)/d-1}]\left(\frac{r}{d}w^{r/d-1})(1+w)^{-(i-1)r}-(i-1)rw^{r/d}(1+w)^{-(i-1)r-1}\right)$$
$$(1-w)^{-d((dn-(d-1)r)/d)}(1+w)^{d(k-1)((dn-(d-1)r)/d)}$$
$$= \frac{r}{(dn-(d-1)r)}[w^{n-r-1}]\left(1-(d(i-1)-1)w\right)(1-w)^{-(dn-(d-1)r)}$$
$$(1+w)^{(k-1)(dn-(d-1)r)-(i-1)r-1}$$
$$= \frac{r}{d(n-r)+r}[w^{n-r}]\left(1-(d(i-1)-1)w\right)$$
$$\sum_{a,b\geq 0}\binom{d(n-r)+r+a-1}{a}\binom{d(k-1)(n-r)+(k-i)r-1}{b}w^{a+b}$$
$$= \frac{r}{d(n-r)+r}\sum_{a\geq 0}\binom{d(n-r)+r+a-1}{a}$$
$$\left[\binom{d(k-1)(n-r)+(k-i)r-1}{n-r-a}-(d(i-1)-1)\binom{d(k-1)(n-r)+(k-i)r-1}{n-r-a-1}\right]$$
$$= \frac{r}{d(n-r)+r}\sum_{a=0}^{n-r}\frac{(k-i)(d(n-r)+r)+ad(i-1)}{d(k-1)(n-r)+(k-i)r}\binom{d(n-r)+r+a-1}{a}$$
$$\binom{d(k-1)(n-r)+(k-i)r}{n-r-a}.$$

The formula is multiplied by $n!$ which is the number of ways of labelling the n nodes. $\square$

The proof of the following corollary follows by setting $r=1$ in Equation (24).

Corollary 11. *The number of labelled d-dimensional k_1-plane trees on n nodes with root labelled i is*

$$\frac{n!}{d(n-1)+1}\sum_{a=0}^{n-1}\frac{(k-i)(d(n-1)+1)+ad(i-1)}{d(k-1)(n-1)+(k-i)}\binom{d(n-1)+a}{a}\binom{d(k-1)(n-1)+(k-i)}{n-a-1}.$$

Corollary 12. *The number of labelled forests of d-dimensional k_1-plane trees on n nodes with r components such that for each component, the root is labelled 1 is given by*

$$\frac{rn!}{d(n-r)+r}\sum_{a=0}^{n-r}\binom{d(n-r)+r+a-1}{a}\binom{(k-1)(d(n-r)+1)}{n-r-a}.$$

Proof. Set $i=1$ in Equation (24). $\square$

On setting $i=k$ in Equation (24), we get the following corollary.

Corollary 13. *The number of labelled forests of d-dimensional k_1-plane trees on n nodes with r components such that for each component, the root is labelled k is given by*

$$\frac{rn!}{d(n-r)+r}\sum_{a=0}^{n-r}\frac{a}{n-r}\binom{d(n-r)+r+a-1}{a}\binom{d(k-1)(n-r)}{n-r-a}.$$

Corollary 14. *The number of labelled forests of k_1-plane trees on n nodes such that* *there are r components such that the root of each component is labelled j is given by*

$$r(n-1)!\sum_{a=0}^{n-r}\frac{(k-i)n+a(i-1)}{(k-1)(n-r)+(k-i)r}\binom{n+a-1}{a}\binom{(k-1)(n-r)+(k-i)r}{n-r-a}.$$

Proof. Set $d=1$ in Equation (24). □

Upon setting $d=2$ in Equation (24), we arrive at the following result.

Corollary 15. *There are*

$$\frac{rn!}{2n-r}\sum_{a=0}^{n-r}\frac{(k-i)(2n-r)+2a(i-1)}{2(k-1)(n-r)+(k-i)r}\binom{2n-r+a-1}{a}\binom{2(k-1)(n-r)+(k-i)r}{n-r-a},$$

labelled forests of k_1-noncrossing trees on n nodes such that there are r components and the root of each component is labelled i.

In the following theorem, we get a more generalized result.

Theorem 9. *There are*

$$\frac{n!}{d(n-r)+r}\sum_{a=0}^{n-r}\frac{(r(k-1)-t)(d(n-r)+r)+adt}{(k-1)(d(n-r)+r)-t}\binom{(k-1)(d(n-r)+r)-t}{n-a-r}\binom{d(n-r)+r+a-1}{a}\binom{r}{r_1,r_2,\ldots,r_{k-i+1}} \quad (25)$$

labelled forests of d-dimensional k_1-plane trees on n nodes such that there are r components, r_j of which have roots labelled j where $j=1,2,\ldots,k-i+1$ and $t:=r_2+2r_3+\cdots+(k-i)r_{k-i+1}$.

Proof. Let $N_i(z)$ be the generating function for d-dimensional k_1-plane trees rooted at a node labelled i, where z marks a node. Since there are r_j d-dimensional k_1-plane trees as part of the forest for $i=1,2,\ldots,k$, then the generating function for the desired forest is $N_1^{r_1}N_2^{r_2}\cdots N_{k-i+1}^{r_{k-i+1}}$. As explained in Section 2, the right substitution to solve the generating function is $N_i(z)=\dfrac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{i-1}}$ where $w=z(1-w)^{-d}(1+w)^{d(k-1)}$. We extract the coefficient z^n in the generating function.

$$\begin{aligned}[z^n]N_1^{r_1}N_2^{r_2}\cdots N_{k-i+1}^{r_{k-i+1}} &= [z^n]((\sqrt[d]{z})^{d-1}\sqrt[d]{w})^{r_1}\cdot\left(\frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{1+w}\right)^{r_2}\cdots\left(\frac{(\sqrt[d]{z})^{d-1}\sqrt[d]{w}}{(1+w)^{k-i}}\right)^{r_{k-i+1}}\\ &= [z^n]z^{(d-1)r/d}w^{r/d}\cdot\left(\frac{1}{1+w}\right)^{r_2}\cdots\left(\frac{1}{(1+w)^{k-i}}\right)^{r_{k-i+1}}\\ &= [z^{n-(d-1)r/d}]w^{r/d}(1+w)^{-t}\end{aligned}$$

where $t:=r_2+2r_3+\cdots+(k-i)r_{k-i+1}$.

By Lagrange-Bürmann inversion, we have

$$
\begin{aligned}
[z^n]N_1^{r_1}N_2^{r_2}\cdots N_{k-i+1}^{r_{k-i+1}} =& \frac{1}{n-(d-1)r/d}[w^{n-(d-1)r/d-1}]\left(\frac{r}{d}w^{r/d-1}(1+w)^{-t} - tw^{r/d}(1+w)^{-t-1}\right) \\
&(1-w)^{-d(n-(d-1)r/d)}(1+w)^{d(k-1)(n-(d-1)r/d)} \\
=& \frac{1}{d(n-r)+r}\left(r[w^{n-r}](1+w)^{(k-1)(d(n-r)+r)-t}\right. \\
&\left. -dt[w^{n-r-1}](1+w)^{(k-1)(d(n-r)+r))-t-1}\right)(1-w)^{-(d(n-r)+r)}.
\end{aligned}
$$

By binomial theorem, we arrive at

$$
\begin{aligned}
[z^n]N_1^{r_1}N_2^{r_2}\cdots N_{k-i+1}^{r_{k-i+1}} &= \frac{1}{d(n-r)+r}\sum_{a\geq 0}\left[r\binom{(k-1)(d(n-r)+r)-t}{n-a-r} - dt\binom{(k-1)(d(n-r)+r)-t-1}{n-a-r-1}\right] \\
&\binom{d(n-r)+r+a-1}{a} \\
&= \frac{1}{d(n-r)+r}\sum_{a=0}^{n-r}\frac{(r(k-1)-t)(d(n-r)+r)+adt}{(k-1)(d(n-r)+r)-t}\binom{(k-1)(d(n-r)+r)-t}{n-a-r}\binom{d(n-r)+r+a-1}{a}.
\end{aligned}
$$

Since there are

$$\binom{r}{r_1, r_2, \ldots, r_{k-i+1}},$$

choices for positions of the trees in the forest then the required formula follows by the product rule of counting. □

If $r_i = r$ in Equation (25) then $t = r(i-1)$ and $r_j = 0$ for all $j \neq i$ so that

$$\frac{n!r}{d(n-r)+r}\sum_{a=0}^{n-r}\frac{(k-i)(d(n-r)+r)+ad(i-1)}{(k-1)(d(n-r)+r)-r(i-1)}\binom{(k-1)(d(n-r)+r)-r(i-1)}{n-a-r}\binom{d(n-r)+r+a-1}{a} \tag{26}$$

counts labelled forests of d-dimensional k_1-plane trees with n nodes and r components such that the roots of all the trees are labelled i. Equation (26) was also obtained in Equation (24).

Corollary 16. *There are*

$$(n-1)!\sum_{a=0}^{n-r}\frac{(r(k-1)-t)n+at}{(k-1)n-t}\binom{(k-1)n-t}{n-a-r}\binom{n+a-1}{a}\binom{r}{r_1, r_2, \ldots, r_{k-i+1}}$$

labelled forests of k_1-plane trees on n nodes such that there are r components, r_j of which have roots labelled j where $j = 1, 2, \ldots, k-i+1$ and $t := r_2 + 2r_3 + \cdots + (k-i)r_{k-i+1}$, [10].

Proof. Set $d = 1$ in Equation (25). □

On setting $d = 2$ in Equation (25), we obtain the following result.

Corollary 17. *There are*

$$\frac{n!}{2n-r}\sum_{a=0}^{n-r}\frac{(r(k-1)-t)(2n-r)+2at}{(k-1)(2n-r)-t}\binom{(k-1)(2n-r)-t}{n-a-r}\binom{2n-r+a-1}{a}\binom{r}{r_1, r_2, \ldots, r_{k-i+1}}$$

labelled forests of k_1-noncrossing trees on n nodes such that there are r components, r_j of which have roots labelled j where $j = 1, 2, \ldots, k - i + 1$ and $t := r_2 + 2r_3 + \cdots + (k - i)r_{k-i+1}$.

7. Conclusion and future work

In this paper, we have enumerated a generalization of a variant of k-plane trees according to the number of nodes, root degree, label of the eldest child of the root being 1, label of the youngest child of the root being $j \neq 1$, the length of the leftmost path and forests. The results obtained in this paper generalize results obtained earlier by Oduol et al. [10, 17]. Equivalent results for generalized k-plane trees have been obtained by Nyariaro et al. in the working paper [19]. It still remains an open problem to enumerate sets of k-plane trees and k-noncrossing trees according to degree sequences, number of leaves, number of endpoints and descents. It would be interesting to enumerate the generalized version of k-plane trees and their variants according to the aforementioned parameters. Asymptotic results for the trees considered in this paper based on the parameters used can also be investigated. Moreover, the study can be extended to enumerate randomized trees. In the future, the applications of the results of this study to algorithm designs and data structure optimization can be explored.

Author contributions: Conceptualization, FOO and IOO; methodology, FOO; software, IOO; validation, FOO, IOO and FON; formal analysis, IOO; investigation, FOO; resources, FON; writing—original draft preparation, FOO; writing—review and editing, FOO; supervision, IOO and FON. All authors have read and agreed to the published version of the manuscript.

Acknowledgments: The first author thanks German Academic Exchange Service (DAAD) for the financial support towards his PhD study at Moi University, Kenya. The second author is grateful to Maseno University for conducive environment to conduct the research.

Institutional review board statement: Not applicable.

Informed consent statement: Not applicable.

Conflict of interest: The authors declare no conflict of interest.

References

1. de Bruijn NG, Morselt BJM. A note on plane trees. Journal of Combinatorial Theory. 1967; 2(1): 27–34. doi: 10.1016/S0021-9800(67)80111-X
2. Noy M. Enumeration of noncrossing trees on a circle. Discrete Mathematics. 1998; 180(1–3): 301–313. doi: 10.1016/S0012-365X(97)00121-0
3. Dershowitz N, Zaks S. Enumerations of ordered trees. Discrete Mathematics. 1980; 31(1): 9–28. doi: 10.1016/0012-365X(80)90168-5
4. Eu SP, Seo S, Shin H. Enumerations of vertices among all rooted ordered trees with levels and degrees. Discrete Mathematics. 2017; 340(9): 2123–2129. doi: 10.1016/j.disc.2017.04.007
5. Du RRX, He J, Yun X. Counting Vertices with Given Outdegree in Plane Trees and k-ary Trees. Graphs and Combinatorics. 2019; 35: 221–229. doi: 10.1007/s00373-018-1975-8

6. Stanley RP. Enumerative Combinatorics, 2nd ed. Cambridge University Press; 1999.
7. Sloane NJA. The On-Line Encyclopaedia of Integer Sequences. Available online: https://oeis.org (accessed on 1 May 2025).
8. Gu NSS, Prodinger H. Bijections for 2-plane trees and ternary trees. European Journal of Combinatorics. 2009; 30(4): 969–985. doi: 10.1016/j.ejc.2008.06.006
9. Gu NSS, Prodinger H, Wagner S. Bijections for a class of labelled plane trees. European Journal of Combinatorics. 2010; 31(3): 720–732. doi: 10.1016/j.ejc.2009.10.007
10. Oduol FO, Okoth IO, Nyamwala FO. Enumeration of a variant of k-plane trees. Journal of Algebra Combinatorics Discrete Structures and Applications. 2024. Preprint.
11. Panholzer A, Prodinger H. Bijection for ternary trees and non-crossing trees. Discrete Mathematics. 2002; 250(1–3): 181–195. doi: 10.1016/S0012-365X(01)00282-5
12. Flajolet P, Noy M. Analytic combinatorics of non-crossing configurations. Discrete Mathematics. 1999; 204(1–3): 203–229. doi: 10.1016/S0012-365X(98)00372-0
13. Hough DS. Descents in noncrossing trees. Electronic Journal of Combinatorics. 2003; 10: 1–5. doi: 10.37236/1753
14. Deutsch E, Noy M. Statistics on non-crossing trees. Discrete Mathematics. 2002; 254(1–3): 75–87. doi: 10.1016/S0012-365X(01)00366-1
15. Yan SHF, Liu X. 2-noncrossing trees and 5-ary trees. Discrete Mathematics. 2009; 309(20): 6135–6138. doi: 10.1016/j.disc.2009.03.044
16. Pang SXM, Lv L. K-noncrossing trees and k-proper trees. In: Proceedings of the 2010 2nd International Conference on Information Engineering and Computer Science; 25-26 December 2010; Wuhan, China. pp. 1–3.
17. Oduol FO, Okoth IO, Nyamwala FO. Enumeration of a variant of k-noncrossing trees. Indian Journal of Discrete Mathematics. 2024. Preprint.
18. Okoth IO, Kasyoki DM. Generalized plane trees. Bulletin of the Institute of Combinatorics and its Applications. 2024. Preprint.
19. Nyariaro AO, Okoth IO, Nyamwala FO. Generalized k-plane trees. Journal of Discrete Mathematics and its Applications. 2024. Preprint.
20. Wilf HS. Generatingfunctionology, 3rd ed. A K Peters/CRC Press; 2006.