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Article

Generating minimal topologies from κ-neighborhoods and primals

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Abstract: This manuscript presents innovative rough approximation operators based on an abstract structure called "minimal topology". This approach offers greater flexibility than traditional topological frameworks by removing the conventional closure requirements for unions and intersections inherent in standard topology, thereby expanding its applicability. We construct eight types of minimal topologies using N_κ-neighborhood systems and the concept of primals. The relationships between these topologies are examined, with a focus on identifying conditions under which they are equivalent. New rough-set models are derived from these minimal topologies, and key properties of their lower and upper approximations are established. Additionally, we apply these approximations to classify subset regions and compute their accuracy measures.

Keywords: approximation spaces; minimal topology; primal; rough set theory
MSC Classification: 03E99; 54A05; 54A10; 54E99

1. Introduction

Many real-world challenges involve uncertainty, including those in fields such as engineering, artificial intelligence, social sciences, and medical sciences. To address these challenges, several mathematical models have been proposed, such as probability theory, fuzzy sets, rough sets, and decision-making frameworks. These models are designed to bridge the gap between classical mathematical methodologies with the uncertainties present in real-world data. However, each of these models has its limitations, which led Pawlak [1] to propose classical rough set theory as a modern tool for handling data imprecision. Rough set theory primarily relies on upper and lower approximations defined through equivalence relations, which, although effective, constrain its applicability. Consequently, researchers have extended rough set theory by incorporating topological concepts to generalize approximations using arbitrary binary relations. Examples of these efforts include using general binary relations [2–6] and neighborhood-rough sets [7–9].

Notable extensions include generalized rough sets [10–12], information systems [13,14], topological structures in rough sets [15–17], and their applications in medical science [18–20].

In 1996, Yao [21] introduced a methodology that inspired further generalizations, incorporating diverse types of relations such as tolerance [13, 22, 23], similarity [24, 25], and general binary relations [26–29]. Additionally, Abd El-Monsef et al. [30] introduced κ-neighborhood spaces (κ-NS) to extend rough set theory through topologies induced by arbitrary relations. This generalization builds upon prior studies [26,31] and has subsequently facilitated a wider range of topological applications within rough set models [32–34], as well as their practical implementations in real-world scenarios [35–37].

Interestingly, the concept of a 'basic-neighborhood' was originally introduced in [29]. Subsequently, works by El-Gayar et al. [35] and Taher et al. [37] utilized κ-NS to define eight types of approximations based on basic-neighborhoods. They provided comprehensive analyses of these relationships, demonstrated novel results, and compared their methods with prior approaches such as those in [21, 27–30, 38]. These studies also highlighted practical applications, particularly in the medical and economic domains.

Mashhour [39] extended the notion of topology to supra-topology by relaxing the condition of finite intersections. A supra-topology on a non-empty set $\mathcal{V}$ is defined as a subclass $\mathcal{S}$ of the power set of $\mathcal{V}$ that satisfies two primary axioms: (1) $\emptyset, \mathcal{V} \in \mathcal{S}$; and (2) $\mathcal{S}$ is closed under arbitrary unions. This flexibility has made supra-topology valuable for modeling real-world problems [40] and for establishing examples that explore relationships between topological concepts.

Minimal spaces, introduced in [41] as a generalization of topological spaces, have proven to be a significant tool in extending and deepening our understanding of concepts in general topology. These spaces offer a broader framework that preserves core topological principles while allowing for greater flexibility in their application and analysis. Additionally, Kuratowski [42] introduced the concept of an ideal as the dual of a filter, which has been further developed in topology and rough set theory. Similarly, the grill [43] was defined, a classical topological construct. Acharjee et al. [44] introduced a primal structure, dual to the grill, and generated primal topologies.

Another notable extension of rough set theory incorporates the concept of ideals [42], which hold significant importance in both topology and rough set methodologies. In topology, ideals are instrumental in defining closure operations, analyzing convergence, and addressing compactification. In the context of rough set theory, they play a critical role in information granulation, influencing the formulation of lower and upper approximations essential for managing data uncertainty [45–48].

In recent developments, Al-Shami and M. Hosny [49] introduced a novel type of neighborhood, termed the $\mathbb{I}_j^{\mathcal{K}}$-neighborhood, building upon the concept of ideals. They developed various rough set approximations based on $\mathbb{I}_j^{\mathcal{K}}$-neighborhoods. This work contained several errors and incorrect results, which have been corrected by R. A. Hosny et al. in [50].

Building on these foundations, this study explores the construction of minimal topologies derived from κ-neighborhood systems and the concept of primals. By leveraging these elements, the study develops eight distinct types of minimal topologies, each defined by specific properties and relationships. These constructions extend the boundaries of classical rough set theory and its intersection with topology, offering a novel perspective on approximation operators.

The primary objective of this work is to establish a unified framework for generating minimal topologies that address the limitations of existing approaches in rough set theory and related fields. The novel operators introduced in this study provide a robust toolset for analyzing complex systems where traditional methods may fall short. Additionally, the manuscript examines the interrelations among the proposed minimal topologies, offering insights into their structural properties and potential applications.

Through this exploration, we aim to enrich the theoretical foundations of minimal topologies and pave the way for their practical implementation in diverse domains, including information systems, decision-making processes, and medical diagnostics. This work contributes to the mathematical understanding of minimal topologies while highlighting their versatility and relevance in solving real-world problems.

2. Fundamentals concepts

The framework broadens the classical notion of topology by relaxing certain restrictions, thereby enabling the study of more diverse and flexible structures.

Let $\mathcal{V}$ be a nonempty finite set (universe), and let $2^{\mathcal{V}}$ denote the collection of all subsets of $\mathcal{V}$. Mashhour [39] extended the classical notion of topology by introducing the concept of supra-topology, which relaxes the requirement of closure under finite intersections. A supra-topology on a nonempty set $\mathcal{V}$ is defined as a subclass Θ of the power set $2^{\mathcal{V}}$ that satisfies the following axioms:

1) $\emptyset, \mathcal{V} \in \Theta$, and
2) Θ is closed under arbitrary unions.

Császár [51] introduced the theory of generalized topological spaces, exploring the fundamental characteristics of these structures. A class $\mathfrak{G} \subseteq 2^{\mathcal{V}}$ is referred to as a generalized topology if it satisfies the following conditions:

1) $\emptyset \in \mathfrak{G}$, and
2) The arbitrary union of elements of $\mathfrak{G}$ also belongs to $\mathfrak{G}$.

A set $\mathcal{V}$ equipped with a generalized topology $\mathfrak{G}$ is called a *generalized topological space* and is denoted by $(\mathcal{V}, \mathfrak{G})$.

In a generalized topological space $(\mathcal{V}, \mathfrak{G})$, the elements of $\mathfrak{G}$ are termed *generalized open sets*, while their complements are referred to as *generalized closed sets*.

A subclass $\mathfrak{M} \subseteq 2^{\mathcal{V}}$ is referred to as a *minimal structure* on $\mathcal{V}$ if $\emptyset, \mathcal{V} \in \mathfrak{M}$. Minimal structures have been primarily studied by Popa and Noiri [52], who also introduced the concepts of $\mathfrak{M}$-open and $\mathfrak{M}$-closed sets. These sets were further characterized using the $\mathfrak{M}$-interior and $\mathfrak{M}$-closure operators, respectively.

Definition 1. *Let $(\mathcal{V}, \mathfrak{M})$ referred to as an minimal space ($\mathfrak{M}$-space)* [52]. *Then, each element of $\mathfrak{M}$ is called $\mathfrak{M}$-open set, and the complement of an $\mathfrak{M}$-open set is referred to as $\mathfrak{M}$-closed set. For a subset H of $\mathcal{V}$, the $\mathfrak{M}$-closure of H denoted by $c_{\mathfrak{M}}(H)$ and the $\mathfrak{M}$-interior of H denoted by $i_{\mathfrak{M}}(H)$, are defined as follows:*

1) $c_{\mathfrak{M}}(H) = \cap\{F : H \subseteq F, \mathcal{V} \setminus F \in \mathfrak{M}\}$*;*
2) $i_{\mathfrak{M}}(H) = \cup\{U \in \mathfrak{M} : U \subseteq H\}$.

Theorem 1 introduces several properties of the interior and closure operators of any subset within a minimal topology, as previously presented in [53].

Theorem 1. *Let $(\mathcal{V}, \mathfrak{M})$ be an $\mathfrak{M}$-space. For subsets $F, H \subseteq \mathcal{V}$, the followings properties hold* [53]*:*

1) $c_{\mathfrak{M}}(\emptyset) = \emptyset$, *and* $i_{\mathfrak{M}}(\mathcal{V}) = \mathcal{V}$*;*
2) $i_{\mathfrak{M}}(H) \subseteq H \subseteq c_{\mathfrak{M}}(H)$*;*
3) $F \subseteq H \Longrightarrow c_{\mathfrak{M}}(F) \subseteq c_{\mathfrak{M}}(H)$, *and* $i_{\mathfrak{M}}(F) \subseteq i_{\mathfrak{M}}(H)$*;*
4) *If* $H \in \mathfrak{M}$, *then* $i_{\mathfrak{M}}(H) = H$*;*
5) *If* $\mathcal{V} \setminus H \in \mathfrak{M}$, *then* $c_{\mathfrak{M}}(H) = H$*;*
6) $c_{\mathfrak{M}}(c_{\mathfrak{M}}(H)) = c_{\mathfrak{M}}(H)$, *and* $i_{\mathfrak{M}}(i_{\mathfrak{M}}(H)) = i_{\mathfrak{M}}(H)$*;*
7) $i_{\mathfrak{M}}(\mathcal{V} \setminus H) = \mathcal{V} \setminus c_{\mathfrak{M}}(H)$, *and* $c_{\mathfrak{M}}(\mathcal{V} \setminus H) = \mathcal{V} \setminus i_{\mathfrak{M}}(H)$.

Csaszar [54] introduced the concept of a weak structure, a mathematical framework that is less restrictive than supra topology, generalized topology, and minimal structure. A weak structure on a nonempty set $\mathcal{V}$ is defined as a subclass $\mho$ of the power set of $\mathcal{V}$ that satisfies the following axiom : $\emptyset \in \mho$.

Definition 2. *Let* $\mathcal{V} \neq \emptyset$. *A class* $\mathcal{P} \subseteq 2^{\mathcal{V}}$ *is named a primal on* $\mathcal{V}$*, if it satisfies the next conditions* [44]*:*

1) $\mathcal{V} \notin \mathcal{P}$*;*
2) $O \notin \mathcal{P}$ *and* $O \subseteq H \Rightarrow H \notin \mathcal{P}$*;*
3) $H \notin \mathcal{P}$ *and* $O \notin \mathcal{P} \Rightarrow H \cap O \notin \mathcal{P}$.

Lemma 1 is significant as it establishes that the third axiom of the primal concept holds in both directions by leveraging the hereditary condition inherent to the same concept.

Lemma 1. *Let* $\mathcal{P}$ *be a primal on* $\mathcal{V}$. *Then,* $H \notin \mathcal{P}$ *and* $O \notin \mathcal{P}$ *if and only if* $H \cap O \notin \mathcal{P}$ [55].

Remark 1. *The class* $\mathcal{P} = \{\emptyset\}$ *did not represent a primal on any set, for instance if* $\mathcal{V} = \{a, b\}$ [55]. *Then* $\{a\} \cap \{b\} = \emptyset \in \mathcal{P}$ *although* $\{a\} \notin \mathcal{P}$ *and* $\{b\} \notin \mathcal{P}$

Remark 2. *If the universe* $\mathcal{V} = \{a\}$*, it is possible to construct an ideal, but not a primal. To construct a primal, the set* $\mathcal{V}$ *must contain at least two distinct elements.*

Remark 3. *The union of two primals results is a primal, whereas the intersection of two primals does not necessarily produce a primal* [44].

Remark 4. *The collection of sets obtained through the intersection (or union) of elements from two primals does not necessarily form a primal on* $\mathcal{V}$*, as demonstrated in the next example* [44]*:*

Example 1. *Let* $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{b, d\}, \{a, d\}, \{a, b, d\}\}$, $\tilde{\mathcal{P}} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$ *be two primals on* $\mathcal{V} = \{a, b, c, d\}$. *Then*

1) $\mathcal{P} \cup \tilde{\mathcal{P}} = 2^{\mathcal{V}} \setminus \{\mathcal{V}, \{a, c, d\}, \{b, c, d\}, \{c, d\}\}$ *is a primal;*
2) *The family* $\nabla = \{A \cap B : A \in \mathcal{P}, B \in \tilde{\mathcal{P}}\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ *is not a primal on* $\mathcal{V}$ *since* $\{a, b, c\} \cap \{a, b, d\} = \{a, b\} \in \nabla$ *but neither* $\{a, b, c\} \in \nabla$ *nor* $\{a, b, d\} \in \nabla$*;*
3) *The family* $\triangle = \{A \cup B : A \in \mathcal{P}, B \in \tilde{\mathcal{P}}\} = 2^{\mathcal{V}}$ *is not a primal on* $\mathcal{V}$*, since* $\mathcal{V} \in \triangle$.

Lemma 2 introduces various types of primals, which will be utilized in the examples presented throughout this paper.

Lemma 2. *Let* $\mathcal{V} \neq \emptyset$ [44,55]. *Then the following families are primals on* $\mathcal{V}$

1) $2^{\mathcal{V}} \setminus \{\mathcal{V}\}$ *(trivial primals);*

2) $\mathcal{P}_y = \{M \subseteq \mathcal{V} : y \notin M\}$ *(excluded point primal);*

3) $\mathcal{P}_O = \{M \subseteq \mathcal{V} : M \cup O \neq \mathcal{V}\}$.

Different sorts of neighborhoods in a set $\mathcal{V}$, defined based on a binary relation $\mathcal{R}$, have been introduced. These neighborhoods are determined by the varied methods in which elements of $\mathcal{V}$ are related to one another according to $\mathcal{R}$. Below is an explanation of these types of neighborhoods:

Definition 3. *Let $\mathcal{R}$ be a binary relation on $\mathcal{V}$* [21,26–28,30]. *The κ-neighborhoods of $y \in \mathcal{V}$ (briefly, $N_\kappa(y)$), for various choices of κ ($\kappa \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$) is defined as follows:*

1) *r-neighborhood: $N_r(y) = \{z \in \mathcal{V} : y\mathcal{R}z\}$;*

2) *l-neighborhood: $N_l(y) = \{z \in \mathcal{V} : z\mathcal{R}y\}$;*

3) *i-neighborhood: $N_i(y) = N_r(y) \cap N_l(y)$;*

4) *u-neighborhood: $N_u(y) = N_r(y) \cup N_l(y)$;*

5) *$\langle r\rangle$-neighborhood: $N_{\langle r\rangle}(y) = \cap\{N_r(z) : y \in N_r(z)\}$ provided that there exists $N_r(z)$ containing y. Otherwise, $N_{\langle r\rangle}(y) = \emptyset$;*

6) *$\langle l\rangle$-neighborhood: $N_{\langle l\rangle}(y) = \cap\{N_l(z) : y \in N_l(z)\}$ provided that there exists $N_l(z)$ containing y. Otherwise, $N_{\langle l\rangle}(y) = \emptyset$;*

7) *$\langle i\rangle$-neighborhood: $N_{\langle i\rangle}(y) = N_{\langle r\rangle}(y) \cap N_{\langle l\rangle}(y)$;*

8) *$\langle u\rangle$-neighborhood: $N_{\langle u\rangle}(y) = N_{\langle r\rangle}(y) \cup N_{\langle l\rangle}(y)$.*

Henceforward, unless otherwise specified, κ will be considered as $\kappa \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$.

Definition 4. *The κ-neighborhoods of a set $H \subseteq \mathcal{V}$ (briefly, $N_\kappa(H)$), for $\kappa \in \{r, l, \langle r\rangle, \langle l\rangle\}$ is defined as $N_\kappa(H) = \cup_{y \in H} N_\kappa(y)$.*

Corollary 1. *Let $H \subseteq \mathcal{V}$. Then,*

1) *i-neighborhood: $N_i(H) = N_r(H) \cap N_l(H)$;*

2) *u-neighborhood: $N_u(H) = N_r(H) \cup N_l(H)$;*

3) *$\langle i\rangle$-neighborhood: $N_{\langle i\rangle}(H) = N_{\langle r\rangle}(H) \cap N_{\langle l\rangle}(H)$;*

4) *$\langle u\rangle$-neighborhood: $N_{\langle u\rangle}(H) = N_{\langle r\rangle}(H) \cup N_{\langle l\rangle}(H)$.*

Definition 5. *Let $\mathcal{R}$ be a binary relation on $\mathcal{V}$, and let ζ_κ: $\mathcal{V} \longrightarrow 2^{\mathcal{V}}$ be a mapping that assigns for each y in $\mathcal{V}$ its κ-neighborhood in $2^{\mathcal{V}}$* [30]. *Then, the triple $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ is referred to as a κ-neighborhood space (κ-NS).*

Proposition 1. *Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS. If $y \in \mathcal{V}$ and $H \subseteq \mathcal{V}$* [19,32,37], *then*

1) *$y \in N_\kappa(y)$, i.e. $N_\kappa(y) \neq \emptyset$, for each κ, if $\mathcal{R}$ is a reflexive relation;*

2) *$H \subseteq N_\kappa(H)$, for each κ, if $\mathcal{R}$ is a reflexive relation;*

3) *$N_{\langle\kappa\rangle}(y) \subseteq N_\kappa(y)$, $\kappa \in \{r, l, i, u\}$, if $\mathcal{R}$ is a reflexive relation;*

4) *$N_r(y) = N_l(y) = N_i(y) = N_u(y)$ and $N_{\langle r\rangle}(y) = N_{\langle l\rangle}(y) = N_{\langle i\rangle}(y) = N_{\langle u\rangle}(y)$, if $\mathcal{R}$ is a symmetric relation;*

5) *$N_{\langle\kappa\rangle}(y) = N_\kappa(y)$, $\kappa \in \{r, l, i, u\}$, if $\mathcal{R}$ is a preorder (reflexive and transitive) relation.*

Theorem 2. *Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS, and let $H \subseteq \mathcal{V}$* [30]. *For each κ, the collection*

$$\tau_\kappa = \{H \subseteq \mathcal{V} : \forall y \in H, N_\kappa(y) \subseteq H\}$$

constitute a topology on $\mathcal{V}$.

A set $H \subseteq \mathcal{V}$ is referred to as a τ_κ-open set if $H \in \tau_\kappa$, while its complement is called a τ_κ-closed set. The family Υ_κ of all τ_κ-closed sets is defined as

$$\Upsilon_\kappa = \{E \subseteq \mathcal{V} : E^c \in \tau_\kappa\},$$

where E^c denotes the complement of E.

The rough approximation operators can be topologically characterized based on Theorem 2, as follows:

Definition 6. *Let τ_κ be a topology on $\mathcal{V}$ generated by κ-NS* [30]. *Then the κ-lower, κ-upper approximations, κ-boundary and κ-accuracy of a subset $H \subseteq \mathcal{V}$ are defined respectively for each κ as:*

1) $\tau_\kappa L(H) = \tau_\kappa int(H)$, *where $\tau_\kappa int(H)$ represents interior of H w.r.t. τ_κ;*
2) $\tau_\kappa U(H) = \tau_\kappa cl(H)$, *where $\tau_\kappa cl(H)$ represents closure of H w.r.t. τ_κ;*
3) $\tau_\kappa \mathcal{B}(H) = \tau_\kappa U(H) - \tau_\kappa L(H)$;
4) $\tau_\kappa \sigma(H) = \frac{|\tau_\kappa L(H)|}{|\tau_\kappa U(H)|}$, *where $|\tau_\kappa U(H)| \neq 0$.*

The triple system $(\mathcal{V}, \mathcal{R}, \mathbb{S}_\kappa)$ is referred to as a κ-supra topological space, where $\mathbb{S}_\kappa$ represents the κ-supra topology on $\mathcal{V}$ generated by Theorem 4. A subset of $\mathcal{V}$ is called a κ-supra open set if it belongs to $\mathbb{S}_\kappa$, and a subset of $\mathcal{V}$ is termed a κ-supra closed set if its complement is an element of $\mathbb{S}_\kappa$. The class of all κ-supra closed subsets of $\mathcal{V}$ is denoted by Π_κ.

3. Generating κ-minimal topological structures from κ-neighborhoods and primals

The extension by primals enriches the structural properties of $\mathbb{S}_\kappa$, providing a broader framework for analyzing and modeling topological systems. This section focuses on the presentation of eight different minimal structures generated from primals and κ-neighborhoods, where $\kappa \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$. The interrelations among these structures will be investigated, with comparative analyses to highlight their distinctions. Furthermore, novel rough approximations derived from these structures will be introduced, and their key properties will be examined.

The significance of Theorem 3 stems from its role in establishing Theorem 4, previously introduced in [56], where researchers incorporated the universal set to derive the super-topology.

Theorem 3. *Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS. Then, for every κ, the class*

$$\mathbb{S}^\circ_\kappa = \{H \subseteq \mathcal{V} : H \subseteq N_\kappa(H)\}$$

is a κ-generalized topology on $\mathcal{V}$.

By incorporating the universal set $\mathcal{V}$ into the collection $\mathbb{S}^\circ_\kappa$, we obtain a κ-supra topology $\mathbb{S}_\kappa$ on $\mathcal{V}$ for every κ, a concept that has been previously introduced and explored in [56].

Theorem 4. *Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS* [56]. *Then for each κ, the class*

$$\mathbb{S}_\kappa = \{\mathcal{V}\} \cup \{H \subseteq \mathcal{V} : H \subseteq N_\kappa(H)\}$$

produces κ-supra topology on $\mathcal{V}$.

For each κ, the $\mathbb{S}_\kappa$-lower, $\mathbb{S}_\kappa$-upper approximations, and $\mathbb{S}_\kappa$-accuracy of a set H are respectively $\mathbb{S}_\kappa L(H) = \mathbb{S}_\kappa int(H)$, $\mathbb{S}_\kappa U(H) = \mathbb{S}_\kappa cl(H)$, $\mathbb{S}_\kappa \sigma(H) = \frac{|\mathbb{S}_\kappa L(H)|}{|\mathbb{S}_\kappa U(H)|}$, where $|\mathbb{S}_\kappa U(H)| \neq 0$ [56].

The significance of Theorem 5 lies in its role in extending Theorem 3 through the application of a primal concept. This leads to the emergence of a new structure, known as the weak structure, which, upon the inclusion of the universal set, results in the minimal topology, as demonstrated in Theorem 6.

Theorem 5. *Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS, and let $\mathcal{P}$ be a primal on $\mathcal{V}$. Then, for every κ, the class*

$$\mathbb{S}_\kappa^{\circ\mathcal{P}} = \{H \subseteq \mathcal{V} : H \setminus N_\kappa(H) \in \mathcal{P}\}$$

is a κ-weak structure on $\mathcal{V}$.

Proof. It is evident that $\emptyset$ belong to $\mathbb{S}_\kappa^{\circ\mathcal{P}}$. □

By including the universal set $\mathcal{V}$ in the collection $\mathbb{S}_\kappa^{\circ\mathcal{P}}$, the resulting structure forms a κ-minimal structure on $\mathcal{V}$.

Theorem 6. *Let $(\mathcal{V}, \mathcal{R}, \zeta_\kappa)$ be a κ-NS, and let $\mathcal{P}$ be a primal on $\mathcal{V}$. Then, for every κ, the class*

$$\mathbb{S}_\kappa^{\mathcal{P}} = \{\mathcal{V}\} \cup \{H \subseteq \mathcal{V} : H \setminus N_\kappa(H) \in \mathcal{P}\}$$

is a κ-minimal structure on $\mathcal{V}$.

Proof. Clearly $\mathcal{V}$ and $\emptyset$ belong to $\mathbb{S}_\kappa^{\mathcal{P}}$. □

The definition of $\mathbb{S}_\kappa^{\mathcal{P}}$ utilizes the interplay between κ-neighborhood systems and the primal $\mathcal{P}$ to establish a structural framework that satisfies the axioms of minimal structures. This approach facilitates the exploration of generalized topological concepts.

The triple system $(\mathcal{V}, \mathcal{R}, \mathbb{S}_\kappa^{\mathcal{P}})$ is referred to as a κ-minimal topological structure (abbreviated as κ-$\mathfrak{M}TS$), where $\mathbb{S}_\kappa^{\mathcal{P}}$ is a κ-minimal topology on $\mathcal{V}$, as constructed in Theorem 6. A subset H of $\mathcal{V}$ is called a κ-minimal open if $H \in \mathbb{S}_\kappa^{\mathcal{P}}$, and it is termed a κ-minimal closed if its complement belongs to $\mathbb{S}_\kappa^{\mathcal{P}}$. The collection of all κ-minimal closed subsets of $\mathcal{V}$ is denoted by $\Upsilon_\kappa^{\mathcal{P}}$.

Example 2. *Let $\mathcal{V} = \{a, b, c, d\}$ and $R = \{(a, b), (a, c), (c, c), (d, b)\}$. If $\mathcal{P} = 2^{\mathcal{V}} \setminus \{\mathcal{V}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}$, then $\mathbb{S}_{\langle i\rangle}^{\mathcal{P}} = \mathbb{S}_{\langle r\rangle}^{\mathcal{P}} = \mathbb{S}_i^{\mathcal{P}} = \mathbb{S}_r^{\mathcal{P}} = \{\emptyset, \mathcal{V}, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}\}$, $\mathbb{S}_u^{\mathcal{P}} = \mathbb{S}_l^{\mathcal{P}} = \{\emptyset, \mathcal{V}, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$, $\mathbb{S}_{\langle u\rangle}^{\mathcal{P}} = \mathbb{S}_{\langle l\rangle}^{\mathcal{P}} = 2^{\mathcal{V}}$.*

It is essential to emphasize that the collection described in Theorem 6 does not necessarily constitute a topology. To substantiate this observation, we present the following remark:

Remark 5. *Example 2 demonstrates that $\mathbb{S}_\kappa^{\mathcal{P}}$ is not necessarily a topology.*

1) $\{a, b\}, \{b, d\} \in \mathbb{S}_r^{\mathcal{P}}$, *but the union* $\{a, b, d\} \notin \mathbb{S}_r^{\mathcal{P}}$*;*
2) $\{a, b, d\}, \{a, c, d\} \in \mathbb{S}_l^{\mathcal{P}}$, *but the intersection* $\{a, d\} \notin \mathbb{S}_l^{\mathcal{P}}$.

Recall that a relation $\mathcal{R}$ is referred to as inverse serial if every element in the set has a nonempty l-neighborhood.

Corollary 2. *If $\mathcal{R}$ is a inverse serial on $\mathcal{V}$, then $\mathbb{S}_r^{\circ\mathcal{P}} = \mathbb{S}_r^{\mathcal{P}}$.*

One of the key findings of this study is the established relationship between the primal structure and the κ-minimal topology, as demonstrated in the following proposition.

Proposition 2. *Let $(\mathcal{V}, \mathcal{R}, \mathbb{S}_\kappa^{\mathcal{P}})$ be a κ-$\mathfrak{M}TS$. Then, for every κ, the inclusion $\mathcal{P} \subseteq \mathbb{S}_\kappa^{\mathcal{P}}$ is satisfied.*

Proof. Let $H \in \mathcal{P}$, then $H \setminus N_\kappa(H) \in \mathcal{P}$. This implies that $H \in \mathbb{S}_\kappa^{\mathcal{P}}$. Consequently, $\mathcal{P} \subseteq \mathbb{S}_\kappa^{\mathcal{P}}$. □

Remark 6. *Example 2 illustrates that $\mathbb{S}_\kappa^{\mathcal{P}} \nsubseteq \mathcal{P}$, for each κ.*

Theorem 7. *Let $(\mathcal{V}, \mathcal{R}, \mathbb{S}_\kappa^{\mathcal{P}})$ be a κ-$\mathfrak{M}TS$, and let $F, H \subseteq \mathcal{V}$. If $F \notin \mathbb{S}_\kappa^{\mathcal{P}}$ and $H \notin \mathbb{S}_\kappa^{\mathcal{P}}$, then $F \cup H \notin \mathbb{S}_\kappa^{\mathcal{P}}$.*

Proof. Suppose $F \notin \mathbb{S}_\kappa^{\mathcal{P}}$ and $H \notin \mathbb{S}_\kappa^{\mathcal{P}}$. This implies that$F, H$ are proper subsets of $\mathcal{V}$ and satisfy $F \setminus N_\kappa(F) \notin \mathcal{P}$ and $H \setminus N_\kappa(H) \notin \mathcal{P}$. By the definition of a primal, it follows that $(F \cap H) \setminus (N_\kappa(F) \cup N_\kappa(H)) \notin \mathcal{P}$, which further implies $(F \cup H) \setminus (N_\kappa(F) \cup N_\kappa(H)) \notin \mathcal{P}$. Using Lemma 1 in [56], we conclude that $(F \cup H) \setminus N_\kappa(F \cup H) \notin \mathcal{P}$. Hence, $F \cup H \notin \mathbb{S}_\kappa^{\mathcal{P}}$. □

Proposition 3. *Let $(\mathcal{V}, \mathcal{R}, \mathbb{S}_\kappa^{\mathcal{P}})$ be a κ-$\mathfrak{M}TS$. Then the following results hold.*

1) $\mathbb{S}_i^{\mathcal{P}} = \mathbb{S}_r^{\mathcal{P}} \cap \mathbb{S}_l^{\mathcal{P}} \subseteq \mathbb{S}_r^{\mathcal{P}} \cup \mathbb{S}_l^{\mathcal{P}} = \mathbb{S}_u^{\mathcal{P}}$.
2) $\mathbb{S}_{\langle i \rangle}^{\mathcal{P}} = \mathbb{S}_{\langle r \rangle}^{\mathcal{P}} \cap \mathbb{S}_{\langle l \rangle}^{\mathcal{P}} \subseteq \mathbb{S}_{\langle r \rangle}^{\mathcal{P}} \cup \mathbb{S}_{\langle l \rangle}^{\mathcal{P}} = \mathbb{S}_{\langle u \rangle}^{\mathcal{P}}$.

Proof. We will provide a proof for the first statement, noting that the second statement can be demonstrated using a similar way.

1) Let $H \in \mathbb{S}_i^{\mathcal{P}}$. By definition, either $H = \mathcal{V}$ or $H \setminus N_i(H) \in \mathcal{P}$. Since $N_i(H) = N_r(H) \cap N_l(H)$, it follows that $H \setminus (N_r(H) \cap N_l(H)) \in \mathcal{P}$. This implies that $H \setminus N_r(H) \in \mathcal{P}$ and $H \setminus N_l(H) \in \mathcal{P}$. Consequently, $H \in \mathbb{S}_r^{\mathcal{P}} \cap \mathbb{S}_l^{\mathcal{P}}$. Clearly, $\mathbb{S}_r^{\mathcal{P}} \cap \mathbb{S}_l^{\mathcal{P}} \subseteq \mathbb{S}_r^{\mathcal{P}} \cup \mathbb{S}_l^{\mathcal{P}}$. Since $\mathbb{S}_u^{\mathcal{P}} = \mathbb{S}_r^{\mathcal{P}} \cup \mathbb{S}_l^{\mathcal{P}}$, the proof is complete. □

Example 2 shows that the converse of Proposition 3 needs not to be true.

Theorem 8. *Let $(\mathcal{V}, \mathcal{R}, \mathbb{S}_\kappa^{\mathcal{P}})$ be a κ-$\mathfrak{M}TS$. Then, for every κ, the inclusion $\mathbb{S}_\kappa \subseteq \mathbb{S}_\kappa^{\mathcal{P}}$ holds.*

Proof. Let $H \in \mathbb{S}_\kappa$, for every κ. By definition, either $H = \mathcal{V}$ or $H \subseteq N_\kappa(H)$. This implies that, $H = \mathcal{V}$ or $H \setminus N_\kappa(H) = \emptyset$. Since $\emptyset \in \mathcal{P}$, it follows that $H \in \mathbb{S}_\kappa^{\mathcal{P}}$. Therefore, we conclude that $\mathbb{S}_\kappa \subseteq \mathbb{S}_\kappa^{\mathcal{P}}$, for every κ.

Remark 7. *According to Theorem 8, the present work serves as a generalization of the results established in* [56]. *However, the converse of Theorem 8 does not hold, as demonstrated by the following example.*

Example 3. *Continued from Example 2.* $\mathbb{S}_{\langle i \rangle} = \mathbb{S}_i = \mathbb{S}_r = \{\emptyset, \mathcal{V}, \{c\}\}$, $\mathbb{S}_l = \{\emptyset, \mathcal{V}, \{c\}, \{a, c\}\}$, $\mathbb{S}_u = \{\emptyset, \mathcal{V}, \{c\}, \{a, b\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$, $\mathbb{S}_{\langle r \rangle} = \{\emptyset, \mathcal{V}, \{b\}, \{c\}, \{b, c\}\}$, $\mathbb{S}_{\langle l \rangle} = \{\emptyset, \mathcal{V}, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$ *and* $\mathbb{S}_{\langle u \rangle} = 2^{\mathcal{V}}$. *Consequently,* $\mathbb{S}_\kappa^{\mathcal{P}} \nsubseteq \mathbb{S}_\kappa$, *for each* κ.

Proposition 4. *Let $\mathcal{P}, \tilde{\mathcal{P}}$ be two primals on $(\mathcal{V}, \mathcal{R})$. If $\mathcal{P} \subseteq \tilde{\mathcal{P}}$, then $\mathbb{S}_\kappa^{\mathcal{P}} \subseteq \mathbb{S}_\kappa^{\tilde{\mathcal{P}}}$, for any κ.*

Proof. Direct to prove. □

Example 4 confirms that the converse of Proposition 4 needs not to be true.

Example 4. *Continued from Example 2. If* $\mathcal{P} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$, $\tilde{\mathcal{P}} = 2^{\mathcal{V}} \setminus \{\mathcal{V}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$ *are two primals on* $\mathcal{V}$. *Suppose* *that* $\kappa = r$. *Then,* $\mathbb{S}_r^{\mathcal{P}} = \{\emptyset, \mathcal{V}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$, *and* $\mathbb{S}_r^{\tilde{\mathcal{P}}} = 2^{\mathcal{V}}$. *Hence,* $\mathbb{S}_r^{\tilde{\mathcal{P}}} \not\subseteq \mathbb{S}_r^{\mathcal{P}}$.

Theorem 9. *Let* $(\mathcal{V}, \mathcal{R}, \mathbb{S}_\kappa^{\mathcal{P}})$ *be a* κ-$\mathfrak{M}TS$. *Then,*

1) $\mathbb{S}_\kappa^{\mathcal{P}}$ *is a discrete topological space, for each* κ, *if* $\mathcal{R}$ *is a reflexive relation on* $\mathcal{V}$*;*
2) $\mathbb{S}_r^{\mathcal{P}} = \mathbb{S}_l^{\mathcal{P}} = \mathbb{S}_i^{\mathcal{P}} = \mathbb{S}_u^{\mathcal{P}}$ *and* $\mathbb{S}_{\langle r\rangle}^{\mathcal{P}} = \mathbb{S}_{\langle l\rangle}^{\mathcal{P}} = \mathbb{S}_{\langle i\rangle}^{\mathcal{P}} = \mathbb{S}_{\langle u\rangle}^{\mathcal{P}}$, *if* $\mathcal{R}$ *is a symmetric relation on* $\mathcal{V}$.

Proof. 1) Since $\mathcal{R}$ is a reflexive relation on $\mathcal{V}$, it follows from Proposition 1 that $H \subseteq N_\kappa(H)$ for each κ. Consequently, $H \in \mathbb{S}_\kappa^{\mathcal{P}}$, indicating that every subset of $\mathcal{V}$ is an element of $\mathbb{S}_\kappa^{\mathcal{P}}$ i.e $\mathbb{S}_\kappa^{\mathcal{P}} = 2^{\mathcal{V}}$. As a result, $\mathbb{S}_\kappa^{\mathcal{P}}$ constitutes the discrete topology on $\mathcal{V}$.

2) According to Proposition 1, the proof is explicit. □

Corollary 3. *Let* $(\mathcal{V}, \mathcal{R}, \mathbb{S}_\kappa^{\mathcal{P}})$ *be a* κ-$\mathfrak{M}TS$. *If* $\mathcal{R}$ *is a reflexive relation on* $\mathcal{V}$, *then* $\mathbb{S}_\kappa^{\mathcal{P}} = \mathbb{S}_\kappa$.

Definition 7. *Let* $\mathbb{S}_\kappa^{\mathcal{P}}$ *be a* κ*-minimal topological space generated by* κ*-NS and a primal* $\mathcal{P}$. *For any subset* $H \subseteq \mathcal{V}$, *the* κ*-minimal interior and* κ*-minimal closure of* H *are defined as follows, respectively:*

$$\mathbb{S}_\kappa^{\mathcal{P}} int(H) = \cup\{G \in \mathbb{S}_\kappa^{\mathcal{P}} : G \subseteq H\},$$
$$\mathbb{S}_\kappa^{\mathcal{P}} cl(H) = \cap\{F \in \Upsilon_\kappa^{\mathcal{P}} : H \subseteq F\}.$$

In the subsequent part, we introduce innovative approximation models founded on the κ-minimal topology, which is constructed using κ-neighborhood systems (κ-NS) and the primal $\mathcal{P}$. Additionally, we explore several key properties of these models and provide an algorithm to demonstrate the computation of $\mathbb{S}_\kappa^{\mathcal{P}}$-accuracy values. To emphasize the importance of these models, we show that they not only enhance the approximation process but also yield accuracy metrics that surpass those of the Al-Shami and Alshammari model [56], regardless of the underlying relation.

Definition 8. *Let* H *be a subset of a* κ-$\mathfrak{M}TS$ $(\mathcal{V}, \mathcal{R}, \mathbb{S}_\kappa^{\mathcal{P}})$. *For each* κ, *the* $\mathbb{S}_\kappa^{\mathcal{P}}$*-lower approximation,* $\mathbb{S}_\kappa^{\mathcal{P}}$*-upper approximation, and* $\mathbb{S}_\kappa^{\mathcal{P}}$*-accuracy of* H *are assigned as follows:*

1) $\mathbb{S}_\kappa^{\mathcal{P}} \underline{L}(H) = \mathbb{S}_\kappa^{\mathcal{P}} int(H)$;
2) $\mathbb{S}_\kappa^{\mathcal{P}} \overline{U}(H) = \mathbb{S}_\kappa^{\mathcal{P}} cl(H)$;
3) $\mathbb{S}_\kappa^{\mathcal{P}} \widehat{\sigma}(H) = \frac{|\mathbb{S}_\kappa^{\mathcal{P}} \underline{L}(H)|}{|\mathbb{S}_\kappa^{\mathcal{P}} \overline{U}(H)|}$, *where* $|\mathbb{S}_\kappa^{\mathcal{P}} \overline{U}(H)| \neq 0$.

It is demonstrable that the accuracy measure $\mathbb{S}_\kappa^{\mathcal{P}} \widehat{\sigma}(H)$ satisfies the condition $0 \leq \mathbb{S}_\kappa^{\mathcal{P}} \widehat{\sigma}(H) \leq 1$. When $\mathbb{S}_\kappa^{\mathcal{P}} \widehat{\sigma}(H)$ approaches 1, it indicates that the $\mathbb{S}_\kappa^{\mathcal{P}}$-lower and $\mathbb{S}_\kappa^{\mathcal{P}}$-upper approximations of H are nearly equal. This implies a higher level of agreement between the approximations, leading to increased accuracy in representing the subset H. If $\mathbb{S}_\kappa^{\mathcal{P}} \widehat{\sigma}(H) = 1$, then H is classified as an $\mathbb{S}_\kappa^{\mathcal{P}}$-exact set. Elsewise, H is termed an $\mathbb{S}_\kappa^{\mathcal{P}}$-rough set, signifying the presence of uncertainty or imprecision in its approximation. The statements in the following proposition illustrate the effectiveness of the proposed approximations in retaining a significant number of properties associated with Pawlak approximations.

Proposition 5. *Let* $\mathbb{S}_\kappa^{\mathcal{P}}$ *be a* κ*-minimal topological space generated by a* κ*-NS and a primal* $\mathcal{P}$. *Then, for* $H, \acute{H} \in 2^{\mathcal{V}}$, *the following holds:*

1) $\mathbb{S}_\kappa^{\mathcal{P}} \underline{L}(\emptyset) = \mathbb{S}_\kappa^{\mathcal{P}} \overline{U}(\emptyset) = \emptyset$ *and* $\mathbb{S}_\kappa^{\mathcal{P}} \underline{L}(\mathcal{V}) = \mathbb{S}_\kappa^{\mathcal{P}} \overline{U}(\mathcal{V}) = \mathcal{V}$;

2) $\mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H) \subseteq H \subseteq \mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H)$;

3) $\mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H) \subseteq \mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(\acute{H})$ *and* $\mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H) \subseteq \mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(\acute{H})$, *if* $H \subseteq \acute{H}$;

4) $\mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H \cap \acute{H}) \subseteq \mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H) \cap \mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(\acute{H})$ *and* $\mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H) \cup \mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(\acute{H}) \subseteq \mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H \cup \acute{H})$;

5) $\mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H) \cup \mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(\acute{H}) \subseteq \mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H \cup \acute{H})$ *and* $\mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H \cap \acute{H}) \subseteq \mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H) \cap \mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(\acute{H})$;

6) $\mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(\mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H)) = \mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H)$ *and* $\mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(\mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H)) = \mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H)$;

7) $\mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H^c) = (\mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H))^c$ *and* $\mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H^c) = (\mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H))^c$.

Proof. Straightforward. □

Remark 8. *It should be noted that the reverse relations of item (5) of the above Proposition need not be true, as illustrated in the application example in Section 4.*

Proposition 6. *Let* H *be a subset of a* κ-$\mathfrak{M}TS$ $(\mathcal{V}, \mathcal{R}, \mathbb{S}_{\kappa}^{\mathcal{P}})$. *For each* κ, *the following statements hold:*

1) $\mathbb{S}_{\kappa}L(H) \subseteq \mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H)$;

2) $\mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H) \subseteq \mathbb{S}_{\kappa}U(H)$;

3) $\mathbb{S}_{\kappa}\sigma(H) \leq \mathbb{S}_{\kappa}^{\mathcal{P}}\widehat{\sigma}(H)$.

Proof. In view of Theorem 8, the proof is clear. □

Proposition 7. *Let* H *be a subset of a* κ-$\mathfrak{M}TS$ $(\mathcal{V}, \mathcal{R}, \mathbb{S}_{\kappa}^{\mathcal{P}})$. *For each* κ, *the following statements hold:*

1) *If* $H \in \mathbb{S}_{\kappa}^{\mathcal{P}}$, *then* $\mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H) = H$;

2) *If* $H \in \Upsilon_{\kappa}^{\mathcal{P}}$, *then* $\mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H) = H$.

Proof. Direct to prove. □

The following propositions are self-evident, and therefore, the proof is omitted.

Proposition 8. *Let* H *be a subset of a* κ-$\mathfrak{M}TS$ $(\mathcal{V}, \mathcal{R}, \mathbb{S}_{\kappa}^{\mathcal{P}})$. *For each* κ, *the following statements hold:*

1) $\mathbb{S}_{i}^{\mathcal{P}}\underline{L}(H) \subseteq \mathbb{S}_{r}^{\mathcal{P}}\underline{L}(H) \subseteq \mathbb{S}_{u}^{\mathcal{P}}\underline{L}(H)$ *and* $\mathbb{S}_{i}^{\mathcal{P}}\underline{L}(H) \subseteq \mathbb{S}_{l}^{\mathcal{P}}\underline{L}(H) \subseteq \mathbb{S}_{u}^{\mathcal{P}}\underline{L}(H)$;

2) $\mathbb{S}_{u}^{\mathcal{P}}\overline{U}(H) \subseteq \mathbb{S}_{r}^{\mathcal{P}}\overline{U}(H) \subseteq \mathbb{S}_{i}^{\mathcal{P}}\overline{U}(H)$ *and* $\mathbb{S}_{u}^{\mathcal{P}}\overline{U}(H) \subseteq \mathbb{S}_{l}^{\mathcal{P}}\overline{U}(H) \subseteq \mathbb{S}_{i}^{\mathcal{P}}\overline{U}(H)$;

3) $\mathbb{S}_{i}^{\mathcal{P}}\widehat{\sigma}(H) \leq \mathbb{S}_{r}^{\mathcal{P}}\widehat{\sigma}(H) \leq \mathbb{S}_{u}^{\mathcal{P}}\widehat{\sigma}(H)$ *and* $\mathbb{S}_{i}^{\mathcal{P}}\widehat{\sigma}(H) \leq \mathbb{S}_{l}^{\mathcal{P}}\widehat{\sigma}(H) \leq \mathbb{S}_{u}^{\mathcal{P}}\widehat{\sigma}(H)$;

4) $\mathbb{S}_{\langle i\rangle}^{\mathcal{P}}\underline{L}(H) \subseteq \mathbb{S}_{\langle r\rangle}^{\mathcal{P}}\underline{L}(H) \subseteq \mathbb{S}_{\langle u\rangle}^{\mathcal{P}}\underline{L}(H)$ *and* $\mathbb{S}_{\langle i\rangle}^{\mathcal{P}}\underline{L}(H) \subseteq \mathbb{S}_{\langle l\rangle}^{\mathcal{P}}\underline{L}(H) \subseteq \mathbb{S}_{\langle u\rangle}^{\mathcal{P}}\underline{L}(H)$;

5) $\mathbb{S}_{\langle u\rangle}^{\mathcal{P}}\overline{U}(H) \subseteq \mathbb{S}_{\langle r\rangle}^{\mathcal{P}}\overline{U}(H) \subseteq \mathbb{S}_{\langle i\rangle}^{\mathcal{P}}\overline{U}(H)$ *and* $\mathbb{S}_{\langle u\rangle}^{\mathcal{P}}\overline{U}(H) \subseteq \mathbb{S}_{\langle l\rangle}^{\mathcal{P}}\overline{U}(H) \subseteq \mathbb{S}_{\langle i\rangle}^{\mathcal{P}}\overline{U}(H)$;

6) $\mathbb{S}_{\langle i\rangle}^{\mathcal{P}}\widehat{\sigma}(H) \leq \mathbb{S}_{\langle r\rangle}^{\mathcal{P}}\widehat{\sigma}(H) \leq \mathbb{S}_{\langle u\rangle}^{\mathcal{P}}\widehat{\sigma}(H)$ *and* $\mathbb{S}_{\langle i\rangle}^{\mathcal{P}}\widehat{\sigma}(H) \leq \mathbb{S}_{\langle l\rangle}^{\mathcal{P}}\widehat{\sigma}(H) \leq \mathbb{S}_{\langle u\rangle}^{\mathcal{P}}\widehat{\sigma}(H)$.

Theorem 10. *Let* $\mathcal{P}, \tilde{\mathcal{P}}$ *be two primals on* $(\mathcal{V}, \mathcal{R})$ *and let* $H \subseteq \mathcal{V}$. *If* $\mathcal{P} \subseteq \tilde{\mathcal{P}}$, *then the following statements hold:*

1) $\mathbb{S}_{\kappa}^{\mathcal{P}}\underline{L}(H) \subseteq \mathbb{S}_{\kappa}^{\tilde{\mathcal{P}}}\underline{L}(H)$;

2) $\mathbb{S}_{\kappa}^{\mathcal{P}}\overline{U}(H) \supseteq \mathbb{S}_{\kappa}^{\tilde{\mathcal{P}}}\overline{U}(H)$;

3) $\mathbb{S}_{\kappa}^{\mathcal{P}}\widehat{\sigma}(H) \leq \mathbb{S}_{\kappa}^{\tilde{\mathcal{P}}}\widehat{\sigma}(H)$.

In Algorithm **Figure 1** and the accompanying Flowchart in **Figure 2**, we detail the procedure for determining whether a subset within a κ-minimal topology is $\mathbb{S}_{\kappa}^{\mathcal{P}}$-exact or $\mathbb{S}_{\kappa}^{\mathcal{P}}$-rough. This method offers a systematic approach to assess the characteristics of subsets based on their alignment with the minimal κ-approximation framework.

Input: Specify a relation $\mathcal{R}$ and a primal $\mathcal{P}$ on the universal set $\mathcal{V}$.
Output: Accurate identification of $\mathbb{s}_{k}^{\mathcal{P}}$-exact and $\mathbb{s}_{k}^{\mathcal{P}}$-rough sets.

Step 1: Initialize the $\boldsymbol{k}$ -Neighborhoods and $\boldsymbol{k}$-Minimal Topological Spaces

1. Specify a relation $\mathcal{R}$ associated with the elements of $\mathcal{V}$.
2. For each k, perform the following:
   - compute k-neighborhoods for each $y \in \mathcal{V}$ as defined in Definition 2.3.
   - compute k-neighborhoods for each $\mathrm{H} \subseteq \mathcal{V}$, as outlined in Definition 2.4 and Corollary 2.1.
   - Use the concept of a primal and the k-neighborhoods of H to construct a minimal topological space for each k, following Theorem 3.4.

Step 2: Calculate the $\mathbb{s}_{k}^{\mathcal{P}}$-Lower Approximation

1. For every $H \subseteq \mathcal{V}$:
   - calculate the $\mathbb{s}_{k}^{\mathcal{P}}$-lower approximation $\mathbb{s}_{k}^{\mathcal{P}}\underline{L}(H)$ using Definition 3.2.
   - If $\mathbb{s}_{k}^{\mathcal{P}}\underline{L}(H), = \emptyset$, conclude that H is an $\mathbb{s}_{k}^{\mathcal{P}}$-rough set.
   - Otherwise, proceed to the next steps.

Step 3: Compute the $\mathbb{s}_{k}^{\mathcal{P}}$-Upper Approximation

1. Calculate the $\mathbb{s}_{k}^{\mathcal{P}}$-upper approximation $\mathbb{s}_{k}^{\mathcal{P}}\overline{\mathrm{U}}(\mathrm{H})$ for all $\mathrm{H} \subseteq \mathcal{V}$, utilizing Definition 3.2.

Step 4: Determine $\mathbb{s}_{k}^{\mathcal{P}}$-accuracy

1. Compute the $\mathbb{s}_{k}^{\mathcal{P}}$-accuracy $\mathbb{s}_{k}^{\mathcal{P}}\hat{\boldsymbol{\sigma}}(\boldsymbol{H}) = \frac{|\mathbb{s}_{k}^{\mathcal{P}}\underline{L}(H)|}{|\mathbb{s}_{k}^{\mathcal{P}}\overline{U}(H)|}$.
2. If $\mathbb{s}_{k}^{\mathcal{P}}\hat{\sigma}(H) = 1$, classify H as an $\mathbb{s}_{k}^{\mathcal{P}}$- exact set.
3. Otherwise, conclude that H is an $\mathbb{s}_{k}^{\mathcal{P}}$-rough set.

Step 5: End.

Figure 1. Algorithm for identifying $\mathbb{S}_{\kappa}^{\mathcal{P}}$-exact and $\mathbb{S}_{\kappa}^{\mathcal{P}}$-rough sets within the framework k-minimal topological structure $(\mathcal{V}, \mathcal{R}, \mathbb{S}_{\kappa}^{\mathcal{P}})$.

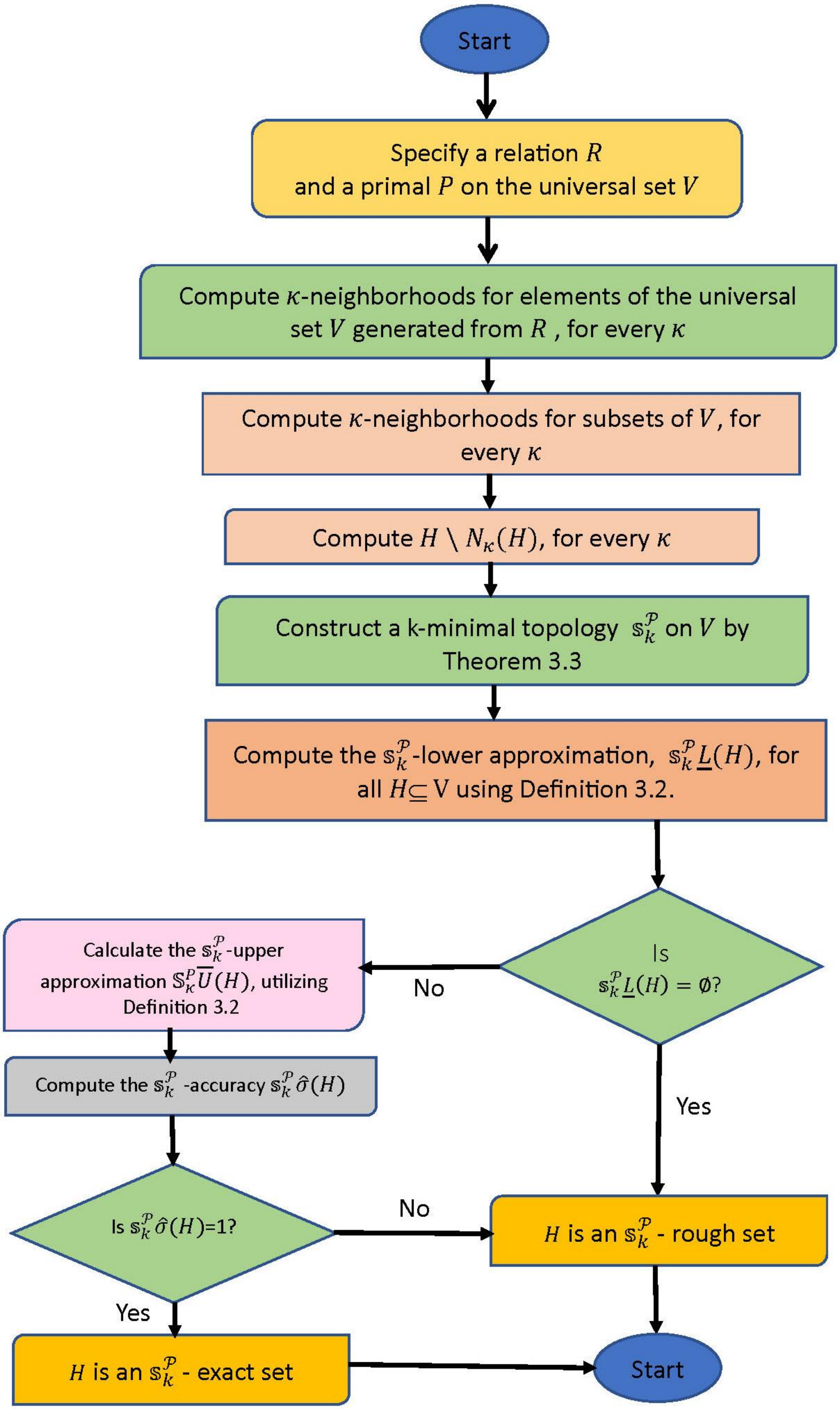

Figure 2. Flowchart for identifying $\mathbb{S}_\kappa^{\mathcal{P}}$-exact and $\mathbb{S}_\kappa^{\mathcal{P}}$-rough subsets in κ-minimal topologies.

4. Dengue fever: A case study on symptom analysis and approximation models

Dengue fever is a significant global health concern, caused by a virus transmitted to humans through infected mosquitoes [57]. Symptoms generally manifest on the third day of infection, with recovery typically occurring within 2 to 7 days. According to the World Health Organization (WHO), this disease has spread to over 120 countries, leading to a significant number of fatalities worldwide, particularly in Asia and South America [58]. Given its global impact, this study employs the proposed approach to analyze dengue fever data.

The dataset, illustrated in **Table 1**, captures critical aspects of the disease. Columns identify dengue fever symptoms : joint and muscle aches (o_1), headache with vomiting (o_2), skin rashes (o_3), and fever (o_4), categorized into three levels: normal (n), high (h), and very high (vh). The decision column (D) indicates whether a patient is infected or not. Rows correspond to the patients under study, denoted as $\mathcal{V} = \{\epsilon_i : i = 1, 2, ..., 8\}$. A check mark $(\checkmark)$ signifies the presence of a symptom, while $(\times)$ indicates its absence.

Table 1. Dengue fever information system.

$\mathcal{V}$	o_1	o_2	o_3	o_4	Dengue fever
ϵ_1	✓	✓	✓	h	✓
ϵ_2	✓	×	×	h	×
ϵ_3	✓	×	×	h	✓
ϵ_4	×	×	×	vh	×
ϵ_5	×	✓	✓	h	×
ϵ_6	✓	✓	×	vh	✓
ϵ_7	✓	✓	×	n	×
ϵ_8	✓	✓	×	vh	✓

4.1. Quantifying symptom similarity

To measure the similarity between patients based on their symptoms, the attributes $\{o_1, o_2, o_3, o_4\}$ are transformed into numerical values reflecting the degree of similarity as shown in **Table 2**. The similarity degree function between two patients c, d, denoted as $\mu(c, d)$, is computed using the formula:

$$\mu(c, d) = \frac{\sum_{i=1}^{m}(a_i(c) = a_i(d))}{m},$$

where m denotes the total number of symptoms (attributes).

Table 2. Similarity degrees between symptoms of patients.

$\mathcal{V}$	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7
ϵ_1	1	0.5	0.5	0	0.75	0.5	0.5
ϵ_2	0.5	1	1	0.5	0.25	0.5	0.5
ϵ_3	0.5	1	1	0.5	0.25	0.5	0.5
ϵ_4	0	0.5	0.5	1	0.25	0.5	0.25
ϵ_5	0.75	0.25	0.25	0.25	1	0.25	0.25
ϵ_6	0.5	0.5	0.5	0.5	0.25	1	0.75
ϵ_7	0.5	0.5	0.5	0.25	0.25	0.75	1

4.2. Defining a relation

A relation $\mathcal{R}$ is proposed based on expert recommendations, defined as follows:

$$(c, d) \in \mathcal{R} \Leftrightarrow 1 > s(c, d) > 0.7,$$

where $\mu(c, d)$ quantifies the ratio of the sum of similar symptoms between c, d relative to the total number of symptoms.

The relation and the threshold value can be adjusted according to system experts' preferences. Notably, the proposed relation $\mathcal{R}$ is symmetric but lacks additional properties. As a result, Pawlak's approximation space is insufficient to effectively describe this system.

4.3. Neighborhood systems and analysis

Using Definition 4, the N_r neighborhood can be computed for each patient ϵ_i, where $i = 1, 2, ..., 8$. Based on these neighborhoods, we construct:

1) The supra topology $\mathbb{S}_r$ The supra topology $\mathbb{S}_r$, which captures broader relationships among patient symptoms.

$\mathbb{S}_r = \{\emptyset, \{\epsilon_1, \epsilon_2\}, \{\epsilon_1, \epsilon_3\}, \{\epsilon_1, \epsilon_5\}, \{\epsilon_1, \epsilon_6\}, \{\epsilon_1, \epsilon_7\}, \{\epsilon_2, \epsilon_4\}, \{\epsilon_2, \epsilon_6\}, \{\epsilon_2, \epsilon_7\}, \{\epsilon_3, \epsilon_4\}, \{\epsilon_3, \epsilon_6\}, \{\epsilon_3, \epsilon_7\}, \{\epsilon_4, \epsilon_6\}, \{\epsilon_6, \epsilon_7\}, \{\epsilon_1, \epsilon_2, \epsilon_3\}, \{\epsilon_1, \epsilon_2, \epsilon_4\}, \{\epsilon_1, \epsilon_2, \epsilon_5\}, \{\epsilon_1, \epsilon_2, \epsilon_6\}, \{\epsilon_1, \epsilon_2, \epsilon_7\}, \{\epsilon_1, \epsilon_3, \epsilon_4\}, \{\epsilon_1, \epsilon_3, \epsilon_5\}, \{\epsilon_1, \epsilon_3, \epsilon_6\}, \{\epsilon_1, \epsilon_3, \epsilon_7\}, \{\epsilon_1, \epsilon_4, \epsilon_6\}, \{\epsilon_1, \epsilon_5, \epsilon_6\}, \{\epsilon_1, \epsilon_5, \epsilon_7\}, \{\epsilon_1, \epsilon_6, \epsilon_7\}, \{\epsilon_2, \epsilon_3, \epsilon_4\}, \{\epsilon_2, \epsilon_3, \epsilon_6\}, \{\epsilon_2, \epsilon_3, \epsilon_7\}, \{\epsilon_2, \epsilon_4, \epsilon_6\}, \{\epsilon_2, \epsilon_4, \epsilon_7\}, \{\epsilon_2, \epsilon_6, \epsilon_7\}, \{\epsilon_3, \epsilon_4, \epsilon_6\}, \{\epsilon_3, \epsilon_4, \epsilon_7\}, \{\epsilon_3, \epsilon_6, \epsilon_7\}, \{\epsilon_4, \epsilon_6, \epsilon_7\}, ..., \mathcal{V}\}$.

2) The minimal topology $\mathbb{S}_r^{\mathcal{P}}$, which integrates the primal structure $\mathcal{P}$ for more refined analyses.

If $\mathcal{P}_{\epsilon_3} = \{M \subseteq \mathcal{V} : \epsilon_3 \notin M\}$ is a primal, then $\mathbb{S}_r^{\mathcal{P}} = \{\emptyset, \{\epsilon_1\}, \{\epsilon_2\}, \{\epsilon_4\}, \{\epsilon_5\}, \{\epsilon_6\}, \{\epsilon_7\}, \{\epsilon_1, \epsilon_2\}, \{\epsilon_1, \epsilon_3\}, \{\epsilon_1, \epsilon_4\}, \{\epsilon_1, \epsilon_5\}, \{\epsilon_1, \epsilon_6\}, \{\epsilon_1, \epsilon_7\}, \{\epsilon_2, \epsilon_4\}, \{\epsilon_2, \epsilon_5\}, \{\epsilon_2, \epsilon_6\}, \{\epsilon_2, \epsilon_7\}, \{\epsilon_3, \epsilon_4\}, \{\epsilon_3, \epsilon_6\}, \{\epsilon_3, \epsilon_7\}, \{\epsilon_4, \epsilon_5\}, \{\epsilon_4, \epsilon_6\}, \{\epsilon_4, \epsilon_7\}, \{\epsilon_5, \epsilon_6\}, \{\epsilon_5, \epsilon_7\}, \{\epsilon_6, \epsilon_7\}, \{\epsilon_1, \epsilon_2, \epsilon_3\}, \{\epsilon_1, \epsilon_2, \epsilon_4\}, \{\epsilon_1, \epsilon_2, \epsilon_5\}, \{\epsilon_1, \epsilon_2, \epsilon_6\}, \{\epsilon_1, \epsilon_2, \epsilon_7\}, \{\epsilon_1, \epsilon_3, \epsilon_4\}, \{\epsilon_1, \epsilon_3, \epsilon_5\}, \{\epsilon_1, \epsilon_3, \epsilon_6\}, \{\epsilon_1, \epsilon_3, \epsilon_7\}, \{\epsilon_1, \epsilon_4, \epsilon_5\}, \{\epsilon_1, \epsilon_4, \epsilon_6\}, \{\epsilon_1, \epsilon_4, \epsilon_7\}, \{\epsilon_1, \epsilon_5, \epsilon_6\}, \{\epsilon_1, \epsilon_5, \epsilon_7\}, \{\epsilon_1, \epsilon_6, \epsilon_7\}, \{\epsilon_2, \epsilon_3, \epsilon_4\}, \{\epsilon_2, \epsilon_3, \epsilon_6\}, \{\epsilon_2, \epsilon_3, \epsilon_7\}, \{\epsilon_2, \epsilon_4, \epsilon_5\}, \{\epsilon_2, \epsilon_4, \epsilon_6\}, \{\epsilon_2, \epsilon_4, \epsilon_7\}, \{\epsilon_2, \epsilon_5, \epsilon_6\}, \{\epsilon_2, \epsilon_5, \epsilon_7\}, \{\epsilon_2, \epsilon_6, \epsilon_7\}, \{\epsilon_3, \epsilon_4, \epsilon_5\}, \{\epsilon_3, \epsilon_4, \epsilon_6\}, \{\epsilon_3, \epsilon_4, \epsilon_7\}, \{\epsilon_3, \epsilon_5, \epsilon_6\}, \{\epsilon_3, \epsilon_5, \epsilon_7\}, \{\epsilon_3, \epsilon_6, \epsilon_7\}, \{\epsilon_4, \epsilon_5, \epsilon_6\}, \{\epsilon_4, \epsilon_5, \epsilon_7\}, \{\epsilon_4, \epsilon_6, \epsilon_7\}, \{\epsilon_5, \epsilon_6, \epsilon_7\}, ..., \mathcal{V}\}$.

It is evident that $\mathbb{S}_r \subseteq \mathbb{S}_r^{\mathcal{P}}$. **Table 3** provides a comparative analysis of the accuracy measures derived from $\mathbb{S}_r^{\mathcal{P}}$ and $\mathbb{S}_r$ for some subsets of $\mathcal{V}$. The results clearly indicate that incorporating the concept of primal structures within $\mathbb{S}_r^{\mathcal{P}}$ leads to a higher degree of accuracy compared to the approximations obtained solely from $\mathbb{S}_r$. This improvement highlights the significance of integrating primal elements into the minimal topological framework to enhance precision and refine the resulting approximations.

Table 3. Comparison between accuracy measures obtained from $\mathbb{S}_r$, and $\mathbb{S}_r^{\mathcal{P}}$ for some subsets of $\mathcal{V}$.

$H \subseteq \mathcal{V}$	$\mathbb{S}_r\sigma(H)$	$\mathbb{S}_r^{\mathcal{P}}\widehat{\sigma}(H)$
$\{\epsilon_1\}$	0	1
$\{\epsilon_2, \epsilon_3\}$	0	0.5
$\{\epsilon_1, \epsilon_3, \epsilon_4\}$	0.75	1
$\{\epsilon_1, \epsilon_3, \epsilon_6\}$	0.75	1
$\{\epsilon_1, \epsilon_4, \epsilon_5\}$	0.66	1
$\{\epsilon_2, \epsilon_4, \epsilon_5, \epsilon_7\}$	0.75	1

5. Conclusion

This study tackles the challenge of managing uncertainty in real-world systems by extending the classical rough set framework. Traditional rough set theory, with its reliance on rigid equivalence relations, often falls short when addressing the imprecision inherent in complex data. By integrating κ-neighborhood systems with primal structures, we have constructed eight novel minimal topologies—parameterized by $\kappa \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$—which provide a more flexible and less restrictive approximation space.

Our theoretical contributions offer enriched structural properties through the extension by primals, enabling a broader framework for analyzing and modeling topological systems. Detailed comparative analyses of these minimal structures reveal their interrelations and distinct characteristics, while the introduction of novel rough approximations underscores the practical potential of our approach.

Furthermore, the application of this framework to dengue fever data demonstrates its versatility. Dengue fever, a critical global health concern with widespread impact in Asia, South America, and beyond, serves as a compelling case study for the proposed methodology. By applying our enhanced approximation techniques, we provide fresh insights into the data patterns associated with the disease, highlighting the framework's capability to handle real-world complexities.

In summary, this work significantly advances the theoretical foundations of rough set theory and opens new avenues for practical applications across diverse fields such as engineering, artificial intelligence, and medical diagnostics.

5.1. Key contributions

The main contributions of this research include:

1) **Novel minimal topologies:** Eight distinct minimal topologies were constructed using N_κ-neighborhood systems and primals. These constructions relax the traditional union and intersection conditions, broadening the applicability of

topological methods.

2) **Interrelations and conditions:** The relationships among these minimal topologies were systematically analyzed, with specific conditions identified under which some topologies coincide.

3) **New rough set models:** Utilizing the developed minimal topologies, innovative rough set models were proposed, providing refined tools for handling uncertainty in data.

4) **Applications and metrics:** The study demonstrated the classification of regions within subsets and computed accuracy measures for these approximations, highlighting the practical utility of the proposed theoretical framework.

5.2. Significance

This research bridges the gap between classical topology and rough set theory, offering a versatile toolset for addressing uncertainties in diverse domains such as artificial intelligence, engineering, and medical sciences. By relaxing traditional constraints, the proposed minimal topologies enable more flexible and efficient modeling of complex systems.

5.3. Future directions

The framework established in this study opens several avenues for further investigation:

1) **Enhanced computational techniques:** Development of efficient algorithms for implementing the proposed rough set models in large-scale datasets.

2) **Integration with other mathematical models:** Exploration of synergies between minimal topologies and alternative uncertainty modeling approaches, such as fuzzy sets and probabilistic models.

3) **Domain-specific applications:** Adaptation and application of the developed models to real-world problems, particularly in fields like bioinformatics, decision-making, and pattern recognition, for instance in the fields [59–61].

4) **Extensions to dynamic systems:** Investigation of the applicability of minimal topologies to evolving systems where the underlying structures change over time.

In conclusion, the minimal topologies introduced in this manuscript offer a robust and adaptable framework for advancing the theoretical and practical dimensions of rough set theory. The findings not only deepen our understanding of topological constructs but also provide a foundation for innovative applications across various disciplines.

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