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Article

A new product for soft sets with its decision-making: Soft lambda-product

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Abstract: Soft sets provide a strong mathematical foundation for managing uncertainty and inventing solutions to parametric data problems. Soft set operations are fundamental elements within soft set theory. In this paper, we introduce a new product operation for soft sets, called the "soft lambda-product," and thoroughly examine its algebraic properties in relation to various types of soft equalities and subsets. By studying the distribution of the soft lambda-product over different soft set operations, we further investigate its relationship with other soft set operations. We conclude with an example demonstrating the method's effectiveness across various applications, employing the *int*-*uni* operator and *int*-*uni* decision function within the soft lambda-product for the *int*-*uni* decision-making method, which identifies an optimal set of elements from available options. This work significantly contributes to the soft set literature, as the theoretical foundations of soft computing methods rely on solid mathematical principles.

Keywords: soft set; soft lambda-product; soft subset; soft equal relations

1. Introduction

George Cantor developed modern set theory, which serves as the basis for all mathematics. Since mathematics demands accuracy in all notions, including sets, ambiguity is one issue associated with the idea of a set. For a long time, mathematicians, logicians, and philosophers have struggled with this ambiguity or representation of incomplete knowledge. In recent years, computer scientists have also been more concerned about it, especially in the field of artificial intelligence. Numerous mathematical techniques, including probability theory, fuzzy set theory [1], and interval mathematics, are available for modeling complex systems; nevertheless, each has its own set of drawbacks. Setting membership values is a known problem in fuzzy set theory, interval mathematics suffers with fluctuating uncertainty, and probability theory only works with stochastically stable systems. Furthermore, the efficiency of these tools is limited by their lack of parameterization, particularly in complicated fields like economics, environmental science, and the social sciences. In 1999, Russian scholar Molodtsov [2] presented soft set theory as a completely general mathematical technique for describing uncertainty. Because there are no rigid restrictions on item descriptions, researchers are free to modify parameters as necessary, which significantly streamlines and improves decision-making, particularly in situations when information is lacking. Soft set theory distinguishes itself by overcoming the challenges and providing a wider range of applications in multidimensional disciplines.

A soft set, which consists of an approximate value set and a predicate, provides a rough description of an item. Although precise answers are required for models in classical mathematics, approximation techniques are employed for complicated models that lack precise solutions. Soft set theory, on the other hand, does not require a precise solution concept because the original description of an item is by nature approximate. Molodtsov [2] showed how versatile soft set theory is by effectively using it in a variety of fields, such as Riemann integration, operations research, game theory, and function smoothness. After Maji et al. [3] applied soft set theory to a decision-making issue for the first time, several researchers [4–10] created early soft set-based decision-making techniques. The "uni-int decision-making" method, a well-known soft set-based technique, was first published by Çağman and Enginoğlu [11]. Later, soft matrix-based decision techniques for the OR, AND, AND-NOT, and OR-NOT operations were presented [12]. Soft set theory has been widely used in decision-making as these techniques have shown to be successful in managing uncertainty and other real-world issues [13–24].

The fundamentals of soft set theory have advanced significantly in the last several years. A thorough theoretical framework including soft subsets, soft set equality, and soft set operations like union, intersection, and AND/OR products was provided by Maji et al. [25]. These ideas were further developed by Pei and Miao [26], who redefined intersection and subset relations and looked at links to information systems. Other operations, such as restricted union, restricted intersection, restricted difference, and extended intersection, were added by Ali et al. [27]. Later works [28–41] addressed conceptual inconsistencies in earlier soft set research, suggested enhancements, and investigated the algebraic structure of soft sets. Whereas Eren and Çalışıcı [42] established a new kind of difference operation for soft sets, Stojanovic [43] investigated the extended symmetric difference of soft sets. Since then, other novel soft set operations have been proposed and examined [44–49].

The fundamental ideas of soft set theory are subsets and soft equality relations. The concept of soft subsets was first put out by Maji et al. [25] and subsequently expanded upon by Pei and Miao [26] and Feng et al. [29]. Two novel kinds of congruence relations and soft equal relations on soft sets were presented by Qin and Hong [50]. Maji's soft distributive laws were modified by Jun and Yang [51], who further extended soft equal relations by using a wider variety of soft subsets. For consistency, J-soft equal relations were established in this study. Noting that not all soft equalities adhere to distributive rules, Liu et al. [52] were motivated by these advancements to propose soft L-subsets and soft L-equal relations.

Building on past work, Feng et al. [53] extended the categories of soft subsets and explored the algebraic aspects of soft product operations, encompassing laws of distribution, commutativity, and association, among other qualities. Using soft L-subsets, they investigated soft products like AND and OR products, looking at these operations under J-equality and L-equality. They also showed that commutative semigroup structures are compatible with soft L-equal relations. See [54–58] for further information on soft equal relations. Molodtsov's initial idea of soft sets was improved by Çağman and Enginoğlu [11], who also established several products in soft set theory, such as *uni-int* decision functions, AND-products, OR-products, AND-NOT-products, and OR-NOT-products. They used these items to propose a systematic decision-making process for choosing the best possibilities among alternatives, offering a real-world illustration of how this strategy may deal with ambiguity. A thorough study of the AND-product was carried out by Sezgin et al. [59], who explored its algebraic characteristics (such as idempotent, commutative, and associative laws) and contrasted them with characteristics of various soft equalities, such as soft F, M, L, and J equalities. It was shown that the set of all soft sets over the universe constitutes a commutative hemiring with identity under soft L-equality when the restricted or extended union is combined with the AND-product. Additionally, they proved that this property also applies when the restricted or extended symmetric difference is paired with the AND-product, forming another commutative hemiring with identity within the framework of soft L-equality.

Çağman and Enginoğlu [11] defined OR-NOT product for soft sets, the domain of the approximation function of which is ExE. They also show that this product is not commutative and associative under M-equality, but holds De Morgan Laws.

This work presents a novel product operation in soft set theory, which we term the "soft lambda-product." Unlike the OR-NOT product for soft sets defined in [11], the domain of the approximation function of soft lambda-product is the cartesian product of the parameter sets' of the soft sets, that is, not ExE. We illustrate this operation with an example and analyze its algebraic features with respect to certain forms of equality and soft subsets, such as M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality. We also look at this product's distributional characteristics over certain different soft set operations. Lastly, we apply the soft decision-making technique to soft lambda-product to choose the best possibilities in a decision-making situation, and we provide an example to show how successful it is. By developing theoretical underpinnings necessary for soft computing applications, this study adds to the body of literature on soft sets. The structure of the paper is as follows: An outline of the main ideas in soft set theory is given in Section 2. In the third section, we present the soft lambda-product and explore its algebraic characteristics in relation to several soft equalities and subsets. The use of *int-uni* and soft lambda-product decision operators in decision-making is examined with a practical example of how this approach may handle uncertainty in Section 4. The last part contains concluding observations.

2. Preliminaries

Definition 1. *[1] Let U be the universal set, E be the parameter, and $P(U)$ be the power set of U and $\mathcal{K} \subseteq E$. A pair $(\mathfrak{J}, \mathcal{K})$ is called a soft set over U where $\mathfrak{J}$ is a set-valued function such that $\mathfrak{J}: \mathcal{K} \rightarrow P(U)$.*

Although Çağman and Enginoğlu [11] modified Molodstov's concept of soft sets, we continue to use the original definition of soft set in our work. Throughout this paper, the collection of all the soft sets defined over U is designated as $\mathrm{S_E(U)}$. Let $\mathcal{K}$ be a fixed subset of E and $\mathrm{S}_{\mathcal{K}}(\mathrm{U})$ be the collection of all those soft sets over U with the fixed parameter set $\mathcal{K}$. That is, while in the set $\mathrm{S}_{\mathcal{K}}(\mathrm{U})$, there are only soft sets whose parameter sets are $\mathcal{K}$; in the set $\mathrm{S_E(U)}$, there are soft sets whose parameter sets may be any set. From now on, for the sake of convenience, soft set(s) will be recognized as SS(s), and parameter set(s) by PS(s).

Definition 2. *[27] Let* $(\mathfrak{J}, \mathcal{K})$ *be an SS over* U. $(\mathfrak{J}, \mathcal{K})$ *is called a relative null SS (with respect to the PS* $\mathcal{K}$*), denoted by* $\emptyset_{\mathcal{K}}$*, if* $\mathfrak{J}(\mathscr{k}) = \emptyset$ *for all* $\mathscr{k} \in \mathcal{K}$ *and* $(\mathfrak{J}, \mathcal{K})$ *is called a relative whole SS (with respect to the PS* $\mathcal{K}$*), denoted by* $U_{\mathcal{K}}$ *if* $\mathfrak{J}(\mathscr{k}) = U$ *for all* $\mathscr{k} \in \mathcal{K}$. *The relative whole SS* U_E *with respect to the universe set of parameters* E *is called the absolute SS over* U

The empty SS over U is the unique SS over U with an empty PS, represented by $\emptyset_{\emptyset}$. Note $\emptyset_{\emptyset}$ and $\emptyset_{\mathcal{M}}$ are different [31]. In the following, we always consider SSs with non-empty PSs in the universe U, unless otherwise stated.

The concept of soft subset, which we refer to here as soft M-subset to prevent confusion, was initially defined by Maji et al. [25] in the following extremely strict way:

Definition 3. *[25] Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be two SSs over* U. $(\mathfrak{J}, \mathcal{K})$ *is called a soft M-subset of* $(\mathfrak{S}, \mathcal{Z})$ *denoted by* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_M (\mathfrak{S}, \mathcal{Z})$ *if* $\mathcal{K} \subseteq \mathcal{Z}$ *and* $\mathfrak{J}(\mathscr{k}) = \mathfrak{S}(\mathscr{k})$ *for all* $\mathscr{k} \in \mathcal{K}$. *Two SSs* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *are said to be soft M-equal, denoted by* $(\mathfrak{J}, \mathcal{K}) =_M (\mathfrak{S}, \mathcal{Z})$ *if* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_M (\mathfrak{S}, \mathcal{Z})$ *and* $(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_M (\mathfrak{J}, \mathcal{K})$.

Definition 4. *[26] Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be two SSs over* U. $(\mathfrak{J}, \mathcal{K})$ *is called a soft F-subset of* $(\mathfrak{S}, \mathcal{Z})$ *denoted by* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{Z})$ *if* $\mathcal{K} \subseteq \mathcal{Z}$ *and* $\mathfrak{J}(\mathscr{k}) \subseteq \mathfrak{S}(\mathscr{k})$ *for all* $\mathscr{k} \in \mathcal{K}$. *Two SSs* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *are said to be soft F-equal, denoted by* $(\mathfrak{J}, \mathcal{K}) =_F (\mathfrak{S}, \mathcal{Z})$ *if* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{Z})$ *and* $(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_F (\mathfrak{J}, \mathcal{K})$.

It is important to note that the definitions of soft F-subset and soft F-equal were originally introduced by Pei and Miao in [26]. However, some papers on soft subsets and soft equalities mistakenly attribute these definitions to Feng et al. in [29]. Consequently, the letter "F" is used to reference this connection.

In Liu et al. [52], it was shown that the soft equality relations =M and =F are equivalent. In other words, $((\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ if and only if $(\mho, \mathcal{M}) =_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$. Since they have the same set of parameters and approximation function, two SSs that satisfy this equivalence are actually identical [52], meaning that $(\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ implies $(\mho, \mathcal{M}) = (\mathfrak{F}, \mathcal{D})$.

Jun and Yang [51] expanded the concepts of F-soft subsets and soft F-equal relations by relaxing the restrictions on parameter sets (PSs). Although Jun and Yang [51] referred to these as the generalized soft subset and generalized soft equal relation we refer to them as soft J-subsets and soft J-equal relations, taking the initial letter of Jun.

Definition 5. *[51] Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be two SSs over* U. $(\mathfrak{J}, \mathcal{K})$ *is called a soft J-subset of* $(\mathfrak{S}, \mathcal{Z})$ *denoted by* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_J (\mathfrak{S}, \mathcal{Z})$ *if for all* $\mathscr{k} \in \mathcal{K}$, *there exists* $z \in \mathcal{Z}$ *such that* $\mathfrak{J}(\mathscr{k}) \subseteq \mathfrak{S}(z)$. *Two SSs* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *are said to be soft J-equal, denoted by* $(\mathfrak{J}, \mathcal{K}) =_J (\mathfrak{S}, \mathcal{Z})$ *if* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_J (\mathfrak{S}, \mathcal{Z})$ *and* $(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_J (\mathfrak{J}, \mathcal{K})$.

It was demonstrated by Liu et al. [52] that $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$, but the converse may not be true.

Liu et al. [52] introduced a new type of soft subset, referred to as soft L-subsets and soft L-equality, which generalizes both soft M-subsets and ontology-based soft subsets. This new concept was inspired by the ideas of soft J-subsets [51] and ontology-based soft subsets [30].

Definition 6. *[52] Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be two SSs over* U. $(\mathfrak{J}, \mathcal{K})$ *is called a soft L-subset of* $(\mathfrak{S}, \mathcal{Z})$ *denoted by* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_L (\mathfrak{S}, \mathcal{Z})$ *if for all* $\mathscr{k} \in \mathcal{K}$, *there exists* $z \in \mathcal{Z}$

such that $\mathfrak{J}(\mathcal{k}) = \mathfrak{S}(z)$. Two SSs $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ are said to be soft J-equal, denoted by $(\mathfrak{J}, \mathcal{K}) =_L (\mathfrak{S}, \mathcal{Z})$ if $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_L (\mathfrak{S}, \mathcal{Z})$ and $(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_L (\mathfrak{J}, \mathcal{K})$.

Concerning the relationships among various types of soft subsets and soft equalities, $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{S}, \mathcal{Z}) \Rightarrow (\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{S}, \mathcal{Z}) \Rightarrow (\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{S}, \mathcal{Z})$ and $(\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} (\mathfrak{S}, \mathcal{Z}) \Rightarrow (\mathfrak{J}, \mathcal{K}) =_{\mathrm{L}} (\mathfrak{S}, \mathcal{Z}) \Rightarrow (\mathfrak{J}, \mathcal{K}) =_{\mathrm{J}} (\mathfrak{S}, \mathcal{Z})$ [52]. However, the converses may not be true. Also, it is well-known that $(\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} (\mathfrak{S}, \mathcal{Z})$ if and only if $(\mathfrak{J}, \mathcal{K}) =_{\mathrm{F}} (\mathfrak{S}, \mathcal{Z})$

We can thus conclude that soft M-equality (and therefore soft F-equality) represents the strictest form of soft equality, while soft J-equality is the weakest. Positioned between these two is the concept of soft L-equality [52].

For further information on soft F-equality, soft M-equality, soft J-equality, soft L-equality, and other definitions of soft subsets and soft equal relations in the literature, please refer to [50–58].

Definition 7. *[27] Let $(\mathfrak{J}, \mathcal{K})$ be an SS over U. The relative complement of $(\mathfrak{J}, \mathcal{K})$, denoted by $(\mathfrak{J}, \mathcal{K})^r$, is defined by $(\mathfrak{J}, \mathcal{K})^r = (\mathfrak{J}^r, \mathcal{K})$, where $\mathfrak{J}^r: \mathcal{K} \to P(U)$ is a mapping given by $\mathfrak{J}^r(\mathcal{k}) = U \setminus \mathfrak{J}(\mathcal{k})$ for all $\mathcal{k} \in \mathcal{K}$. From now on, $U \setminus \mathfrak{J}(\mathcal{k}) = [\mathfrak{J}(\mathcal{k})]'$ is designated by $\mathfrak{J}'(\mathcal{k})$ for the sake of designation.*

Definition 8. *[25] Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be two SSs over U. The AND-product ($\wedge$-product) of $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$, denoted by $(\mathfrak{J}, \mathcal{K}) \Lambda (\mathfrak{S}, \mathcal{Z})$, is defined by $(\mathfrak{J}, \mathcal{K}) \Lambda (\mathfrak{S}, \mathcal{Z}) = (\mathfrak{Q}, \mathcal{K}x\mathcal{Z})$, where for all $(\mathcal{k}, z) \in \mathcal{K}x\mathcal{Z}$, $\mathfrak{Q}(\mathcal{k}, z) = \mathfrak{J}(\mathcal{k}) \cap \mathfrak{S}(z)$.*

Definition 9. *[25] Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be two SSs over U. The OR-product ($\vee$-product) of $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$, denoted by $(\mathfrak{J}, \mathcal{K}) \vee (\mathfrak{S}, \mathcal{Z})$, and is defined by $(\mathfrak{J}, \mathcal{K}) \vee (\mathfrak{S}, \mathcal{Z}) = (\mathfrak{Q}, \mathcal{K}x\mathcal{Z})$, where for all $(\mathcal{k}, z) \in \mathcal{K}x\mathcal{Z}$, $\mathfrak{Q}(\mathcal{k}, z) = \mathfrak{J}(\mathcal{k}) \cup \mathfrak{S}(z)$*

Çağman [60] presented the notions of inclusive complement and exclusive complement as new ideas in set theory and used comparison to study their links. In [60], these novel ideas were also used in group theory. Some novel complements were introduced by Sezgin et al. [61], who also looked into their relationships and applied them to group theory.

Definition 10. *[61] Let A and B be two subsets of the universe. Then, A lambda B is defined by $A \lambda B := A \cup B'$.*

Subsequently, the lambda operation was applied to SS theory to introduce new SS operations [62–64]. Let "$\odot$" represent set operations such as $\cap, \cup, \setminus, \triangle$. The following definitions are provided for restricted, extended, and soft binary piecewise operations.

Definition 11. *[27] Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be SSs over U. The restricted $\odot$ operation of $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$, denoted by $(\mathfrak{J}, \mathcal{K}) \odot_R (\mathfrak{S}, \mathcal{Z})$ is defined by $(\mathfrak{J}, \mathcal{K}) \odot_R (\mathfrak{S}, \mathcal{Z}) = (\mathfrak{Q}, \mathcal{C})$, where $\mathcal{C} = \mathcal{K} \cap \mathcal{Z}$ and if $\mathcal{C} \neq \emptyset$, then for all $c \in \mathcal{C}$, $\mathfrak{Q}(c) = \mathfrak{J}(c) \odot \mathfrak{S}(c)$; if $\mathcal{C} = \emptyset$, then $(\mathfrak{J}, \mathcal{K}) \odot_R (\mathfrak{S}, \mathcal{Z}) = \emptyset_\emptyset$.*

Definition 12. *[27,43,62] Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be SSs over U. The extended $\odot$ operation of $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$, denoted by $(\mathfrak{J}, \mathcal{K}) \odot_\varepsilon (\mathfrak{S}, \mathcal{Z})$ is defined by $(\mathfrak{J}, \mathcal{K}) \odot_\varepsilon (\mathfrak{S}, \mathcal{Z}) = (\mathfrak{Q}, \mathcal{C})$, where $\mathcal{C} = \mathcal{K} \cup \mathcal{Z}$ and for all $c \in \mathcal{C}$,*

$$\mathfrak{Q}(c) = \begin{cases} \mathfrak{J}(c), & c \in \mathcal{K} \setminus \mathcal{Z} \\ \mathfrak{S}(c), & c \in \mathcal{Z} \setminus \mathcal{K} \\ \mathfrak{J}(c) \odot \mathfrak{S}(c), & c \in \mathcal{K} \cap \mathcal{Z} \end{cases}.$$

Definition 13. *[44,49,64] Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. The soft binary piecewise $\odot$ operation of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K})\ \widetilde{\widetilde{\odot}}\ (\mathfrak{S},\mathcal{Z})$ is defined by $(\mathfrak{J},\mathcal{K})\ \widetilde{\widetilde{\odot}}\ (\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K})$, where for all $c \in \mathcal{K}$,*

$$\mathfrak{Q}(c) = \begin{cases} \mathfrak{J}(c), & c \in \mathcal{K} \setminus \mathcal{Z} \\ \mathfrak{J}(c) \odot \mathfrak{S}(c), & c \in \mathcal{K} \cap \mathcal{Z} \end{cases}.$$

For more about the soft algebraic structures of SSs, hypersoft sets, picture fuzzy soft sets, we refer to [65–89].

3. Soft lambda-product and its algebraic properties

We proposed the soft lambda-product, a novel product for SSs, in this part. We provide an example and analyze its algebraic characteristics in depth with respect to specific kinds of soft equalities and soft subsets.

Definition 14. *Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. The soft lambda-product of $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$, denoted by $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})$, is defined by $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}\times\mathcal{Z})$, where for all $(k,z) \in \mathcal{K}x\mathcal{Z}$,*

$$\mathfrak{Q}(k,z) = \mathfrak{J}(k)\lambda\mathfrak{S}(z).$$

Here, $\mathfrak{J}(k)\lambda\mathfrak{S}(z) = \mathfrak{J}(k) \cup \mathfrak{S}'(z)$.

Here note that Çağman and Enginoğlu [11] defined $OR-NOT-$product for SSs in a similar way to soft lambda-product as follows:

Definition 15. *[11] Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over U. The $\overline{\nabla}$-product ($OR-NOT-$product) of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$, denoted by $(\mho,\mathcal{M})\ \overline{\nabla}\ (\mathfrak{F},\mathcal{D})$, is defined by $(\mho,\mathcal{M})\ \overline{\nabla}\ (\mathfrak{F},\mathcal{D}) = (\mathcal{O},ExE)$, where for all $(m,d) \in ExE$, $\mathcal{O}(m,d) = \mho(m) \cup \mathfrak{F}'(d)$.*

It is observed that while the domain of the approximation function of $OR-NOT-$product of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ is ExE, the domain of the approximation function of soft lambda-product of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ is $\mathcal{M}$x$\mathcal{D}$.

Example 1. *Let $E = \{\ell_1,\ell_2,\ell_3,\ell_4\}$ be the PS, $\mathcal{K} = \{\ell_2,\ell_3\}$, and $\mathcal{Z} = \{\ell_2,\ell_4\}$ be the subsets of E, $U = \{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5\}$ be the universal set, $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U such that*

$$(\mathfrak{J},\mathcal{K}) = \{(\ell_2,\{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5\}),(\ell_3,\{\hbar_3,\hbar_5\})\}$$

$$(\mathfrak{S},\mathcal{Z}) = \{(\ell_2,\{\hbar_1,\hbar_2,\hbar_3\})(\ell_4,\{\hbar_2,\hbar_3,\hbar_4\})\}.$$

Let $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) = (\mathfrak{Q},\mathcal{K}x\mathcal{Z})$. Then,

$$(\mathfrak{Q},\mathcal{K}\times\mathcal{Z}) = \{((\ell_2,\ell_2),\{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5\}),((\ell_2,\ell_4),\{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5\}),((\ell_3,\ell_2),\{\hbar_3,\hbar_4,\hbar_5\}),((\ell_3,\ell_4),\{\hbar_1,\hbar_3,\hbar_5\})\}.$$

Since it is more practical than writing in the list method style, the **Table 1** method can be applied here:

Table 1. The table designation of the soft lambda-product's result of the soft sets in Example 1.

$(\mathfrak{J},\mathcal{K})\ \Lambda_\lambda\ (\mathfrak{S},\mathcal{Z})$	ℓ_2	ℓ_4
ℓ_2	$\{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5\}$	$\{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5\}$
ℓ_3	$\{\hbar_3,\hbar_4,\hbar_5\}$	$\{\hbar_1,\hbar_3,\hbar_5\}$

Proposition 1. *V_λ-product is closed in $S_E(U)$.*
Proof 1. It is clear that V_λ-product in a binary operation in $S_E(U)$. In fact, let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. Then,

$$V_\lambda : S_E(U)\times S_E(U)\to S_E(U)\big((\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{Z})\big)\to(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})=(\mathfrak{Q},\mathcal{K}\mathrm{x}\mathcal{Z})=(\mathfrak{Q},\mathcal{C}).$$

That is, $(\mathfrak{Q},\mathcal{C})$ is an SS over U, since the set $S_E(U)$ contains all the SS over U. Here, note that the set $S_{\mathcal{K}}(U)$ is not closed under V_λ-product, since if $(\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{K})$ are the elements of $S_{\mathcal{K}}(U)$, $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{K})$ is an element of $S_{\mathcal{K}\mathrm{x}\mathcal{K}}(U)$, not $S_{\mathcal{K}}(U)$. □
Proposition 2. *Let $(\mathfrak{J},\mathcal{K})$, $(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ be SSs over U. Then,*

$$(\mathfrak{J},\mathcal{K})V_\lambda[(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{Q},\mathcal{C})]\neq_M[(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})]V_\lambda(\mathfrak{Q},\mathcal{C}).$$

That is, V_λ-product is not associative in $S_E(U)$.
Proof 2. We provided an example to show that V_λ-product is not associative in $S_E(U)$. Let $E=\{\ell_1,\ell_2,\ell_3,\ell_4\}$ be PS, $\mathcal{K}=\{\ell_2,\ell_3\}$, $\mathcal{Z}=\{\ell_1\}$ and $\mathcal{C}=\{\ell_4\}$ be the subsets of E, $U=\{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5\}$ be the universal set, $(\mathfrak{J},\mathcal{K})$, $(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ be SSs over üzere U such that $(\mathfrak{J},\mathcal{K})=\{(\ell_2,\{\hbar_3,\hbar_4\}),(\ell_3,\{\hbar_1\})\}$, $(\mathfrak{S},\mathcal{Z})=\{(\ell_1,\emptyset)\}$ and $(\mathfrak{Q},\mathcal{C})=\{(\ell_4,\{\hbar_1,\hbar_3,\hbar_5\})\}$. We show that

$$(\mathfrak{J},\mathcal{K})V_\lambda[(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{Q},\mathcal{C})]\neq_M[(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})]V_\lambda(\mathfrak{Q},\mathcal{C}).$$

Let $(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{Q},\mathcal{C})=(\zeta,\mathcal{Z}\times\mathrm{C})$. Then,

$$(\zeta,\mathcal{Z}\times\mathrm{C})=\big\{\big((\ell_1,\ell_4),\{\hbar_2,\hbar_4\}\big)\big\}.$$

Assume that $(\mathfrak{J},\mathcal{K})V_\lambda(\zeta,\mathcal{Z}\times\mathrm{C})=\big(\mathfrak{E},\mathcal{K}\times(\mathcal{Z}\times\mathrm{C})\big)$. Thus,

$$\big(\mathfrak{E},\mathcal{K}\times(\mathcal{Z}\times\mathrm{C})\big)=\Big\{\Big(\big(\ell_2,(\ell_1,\ell_4)\big),\{\hbar_1,\hbar_3,\hbar_4,\hbar_5\}\Big),\Big(\big(\ell_3,(\ell_1,\ell_4)\big),(\hbar_1,\hbar_3,\hbar_5)\Big)\Big\}.$$

Let $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})=(\mathfrak{G},\mathcal{K}\times\mathcal{Z})$. Thereby,

$$(\mathfrak{G},\mathcal{K}\times\mathcal{Z})=\big\{\big((\ell_2,\ell_1),\{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5\}\big),\big((\ell_3,\ell_1),\{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5\}\big)\big\}.$$

Suppose that $(\mathfrak{G},\mathcal{K}\times\mathcal{Z})V_\lambda(\mathfrak{Q},\mathcal{C})=(\zeta,(\mathcal{K}\times\mathcal{Z})\times\mathcal{C})$. Hence,

$$(\zeta,(\mathcal{K}\times\mathcal{Z})\times\mathcal{C})=\Big\{\Big(\big((\ell_2,\ell_1),\ell_4\big),\{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5\}\Big),\Big(\big((\ell_3,\ell_1),\ell_4\big),\{\hbar_1,\hbar_2,\hbar_3,\hbar_4,\hbar_5\}\Big)\Big\}.$$

Thus, $\big(\mathfrak{E},\mathcal{K}\times(\mathcal{Z}\times\mathrm{C})\big)\neq_M(\zeta,(\mathcal{K}\times\mathcal{Z})\times\mathcal{C})$. Similarly, $\big(\mathfrak{E},\mathcal{K}\times(\mathcal{Z}\times\mathrm{C})\big)\neq_L(\zeta,(\mathcal{K}\times\mathcal{Z})\times\mathcal{C})$ and $\big(\mathfrak{E},\mathcal{K}\times(\mathcal{Z}\times\mathrm{C})\big)\neq_J(\zeta,(\mathcal{K}\times\mathcal{Z})\times\mathcal{C})$. □
Proposition 3. *Let $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{S},\mathcal{Z})$ be SSs over U. Then, $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})\neq_M(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})$. Namely, V_λ-product is not commutative in $S_E(U)$.*
Proof 3. Let $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})=(\mathfrak{Q},\mathcal{K}\times\mathcal{Z})$ and $(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})=(\mathfrak{G},\mathcal{Z}\times\mathcal{K})$. Since $\mathcal{K}\times\mathcal{Z}\neq\mathcal{Z}\times\mathcal{K}$, the rest of the proof is obvious. □

Proposition 4. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) \neq_J (\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})$. *That is,* V_λ*-product is not commutative in* $S_E(U)$ *under J-equality.*

Proof 4. We provided an example to show that V_λ-product is not commutative under J-equality in $S_E(U)$. Let $E = \{\ell_1, \ell_2, \ell_3, \ell_4\}$ be the PS, $\mathcal{K} = \{\ell_2, \ell_3\}$, and $\mathcal{Z} = \{\ell_1\}$ be the subsets of E, $U = \{\not{f}_1, \not{f}_2, \not{f}_3, \not{f}_4, \not{f}_5\}$ be the universal set and $(\mathfrak{J},\mathcal{K})$, and $(\mathfrak{S},\mathcal{Z})$ be SSs over U such that $(\mathfrak{J},\mathcal{K}) = \{(\ell_2, \{\not{f}_3, \not{f}_4\}), (\ell_3, \{\not{f}_1\})\}$, $(\mathfrak{S},\mathcal{Z}) = \{(\ell_1, \emptyset)\}$. We show that

$$(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) \neq_J (\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K}).$$

Let $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) = (\mathcal{W}, \mathcal{K}\times\mathcal{Z})$, where

$$(\mathcal{W}, \mathcal{K}\times\mathcal{Z}) = \{((\ell_2,\ell_1), \{\not{f}_1, \not{f}_2, \not{f}_3, \not{f}_4, \not{f}_5\}), ((\ell_3,\ell_1), \{\not{f}_1, \not{f}_2, \not{f}_3, \not{f}_4, \not{f}_5\})\}.$$

Assume that $(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K}) = (\mathcal{H}, \mathcal{Z}\times\mathcal{K})$, where

$$(\mathcal{H}, \mathcal{Z}\times\mathcal{K}) = \Big(((\ell_1,\ell_2), \{\not{f}_1, \not{f}_2, \not{f}_5\})((\ell_1,\ell_3), \{\not{f}_2, \not{f}_3, \not{f}_4, \not{f}_5\})\Big).$$

Hence, $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) \neq_J (\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})$. Moreover, $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) \neq_L (\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})$. □

Proposition 5. *Let* $(\mathfrak{J},\mathcal{K})$ *be an SS over* U. *Then,* $(\mathfrak{J},\mathcal{K})V_\lambda\emptyset_\emptyset =_M \emptyset_\emptyset V_\lambda(\mathfrak{J},\mathcal{K}) =_M \emptyset_\emptyset$. *Namely,* $\emptyset_\emptyset$*-the empty SS-is the absorbing element of* V_λ*-product in* $S_E(U)$.

Proof 5. Let $\emptyset_\emptyset = (\mathfrak{Q}, \emptyset)$ and $(\mathfrak{J},\mathcal{K})V_\lambda\emptyset_\emptyset = (\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\emptyset) = (\mathfrak{S}, \mathcal{K}\times\emptyset) = (\mathfrak{S},\emptyset)$. Since $\emptyset_\emptyset$ is the only SS whose PS is $\emptyset$, $(\mathfrak{S},\emptyset) = \emptyset_\emptyset$ is obtained. Similarly, $\emptyset_\emptyset V_\lambda(\mathfrak{J},\mathcal{K}) =_M \emptyset_\emptyset$. □

Proposition 6. *Let* $(\mho,\mathcal{K})$ *be an SS over* U. *Then,* $U_\mathcal{K}V_\lambda(\mho,\mathcal{K}) =_L U_\mathcal{K}$. *That is,* $U_\mathcal{K}$ *is the left absorbing element of* $\vee_\lambda$*-product in* $S_\mathcal{K}(U)$ *under L-equality.*

Proof 6. Let $U_\mathcal{K} = (\Upsilon, \mathcal{K})$ and $(\Upsilon,\mathcal{K})V_\lambda(\mho,\mathcal{K}) = (\eth, \mathcal{K}x\mathcal{K})$. Then, for all $\hbar \in \mathcal{K}$, $\Upsilon(\hbar) = U$ and for all $(\hbar, z) \in \mathcal{K}x\mathcal{K}$, $\eth(\hbar,z) = \Upsilon(\hbar) \cup \mho'(z) = U \cup \mho'(z) = U$. Since, for all $(\hbar, z) \in \mathcal{K}x\mathcal{K}$, there exists $\hbar \in \mathcal{K}$ such that $\eth(\hbar,z) = U = \Upsilon(\hbar)$, implying that $U_\mathcal{K}V_\lambda(\mho,\mathcal{K}) \cong\!\!\subseteq_L U_\mathcal{K}$. Moreover, for all $\hbar \in \mathcal{K}$, there exists $(\hbar, z) \in \mathcal{K}x\mathcal{K}$ such that $\Upsilon(\hbar) = U = \eth(\hbar,z)$, implying that $U_\mathcal{K} \cong\!\!\subseteq_L U_\mathcal{K}V_\lambda(\mho,\mathcal{K})$. Thereby, $U_\mathcal{K}V_\lambda(\mho,\mathcal{K}) =_L U_\mathcal{K}$. □

Proposition 7. *Let* $(\mho,\mathcal{K})$ *be an SS over* U. *Then,* $(\mho,\mathcal{K})V_\lambda U_\mathcal{K} =_L (\mho,\mathcal{K})$. *That is,* $U_\mathcal{K}$ *is the right identity element of* $\vee_\lambda$*-product in* $S_\mathcal{K}(U)$ *under L-equality.*

Proof 7. Let $U_\mathcal{K} = (\Upsilon, \mathcal{K})$ and $(\mho,\mathcal{K})V_\lambda(\Upsilon,\mathcal{K}) = (\eth, \mathcal{K}x\mathcal{K})$. Then, for all $\hbar \in \mathcal{K}$, $\Upsilon(\hbar) = U$ and for all $(\hbar, z) \in \mathcal{K}x\mathcal{K}$, $\eth(\hbar,z) = \mho(\hbar) \cup \Upsilon'(z) = \mho(\hbar) \cup \emptyset = \mho(\hbar)$. Since, for all $(\hbar, z) \in \mathcal{K}x\mathcal{K}$, there exists $\hbar \in \mathcal{K}$ such that $\eth(\hbar,z) = \mho(\hbar)$, implying that $(\mho,\mathcal{K})V_\lambda U_\mathcal{K} \cong\!\!\subseteq_L (\mho,\mathcal{K})$. Moreover, for all $\hbar \in \mathcal{K}$, there exists $(\hbar, z) \in \mathcal{K}x\mathcal{K}$ such that $\mho(\hbar) = \eth(\hbar,z)$, implying that $(\mho,\mathcal{K}) \cong\!\!\subseteq_L (\mho,\mathcal{K})V_\lambda U_\mathcal{K}$. Thereby, $(\mho,\mathcal{K})V_\lambda U_\mathcal{K} =_L (\mho,\mathcal{K})$. □

Proposition 8. *Let* $(\mathfrak{J},\mathcal{K})$ *be an SS over* U. *Then,* $(\mathfrak{J},\mathcal{K})V_\lambda\emptyset_\mathcal{K} =_M U_{\mathcal{K}\times\mathcal{K}}$ *and* $\emptyset_\mathcal{K}V_\lambda(\mathfrak{J},\mathcal{K}) =_M (\mathfrak{J}, \mathcal{K}\times\mathcal{K})^r$.

Proof 8. Let $\emptyset_\mathcal{K} = (\mathfrak{Q}, \mathcal{K})$, where for all $\hbar \in \mathcal{K}$, $\mathfrak{Q}(\hbar) = \emptyset$. Assume that $(\mathfrak{J},\mathcal{K})V_\lambda\emptyset_\mathcal{K} = (\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\mathcal{K}) = (\mathfrak{S}, \mathcal{K}\times\mathcal{K})$, where for all $(\hbar, z) \in \mathcal{K}\times\mathcal{K}$,

$\mathfrak{S}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{Q}'(z) = \mathfrak{J}(\mathscr{k}) \cup \emptyset' = \mathfrak{J}(\mathscr{k}) \cup \mathrm{U} = \mathrm{U}$, implying that $(\mathfrak{S}, \mathcal{K} \times \mathcal{K}) = (\mathrm{U}, \mathcal{K} \times \mathcal{K})$. Let $\emptyset_{\mathcal{K}} \mathrm{V}_\lambda (\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} (\mathfrak{N}, \mathcal{K} \times \mathcal{K})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{K}$, $\mathfrak{N}(\mathscr{k}, z) = \mathfrak{Q}(\mathscr{k}) \cup \mathfrak{J}'(z) = \emptyset \cup \mathfrak{J}'(z) = \mathfrak{J}'(z)$. Thus, $(\mathfrak{N}, \mathcal{K} \times \mathcal{K}) = (\mathfrak{J}, \mathcal{K} \times \mathcal{K})^{\mathrm{r}}$. □

Proposition 9. *Let* $(\mathfrak{J}, \mathcal{K})$ *be an SS over* U. *Then,* $(\mathfrak{J}, \mathcal{K}) V_\lambda U_{\mathcal{K}} =_M (\mathfrak{J}, \mathcal{K})$ *and* $U_{\mathcal{K}} V_\lambda (\mathfrak{J}, \mathcal{K}) =_M U_{\mathcal{K} \times \mathcal{K}}$.

Proof 9. Let $\mathrm{U}_{\mathcal{K}} = (\mathfrak{Q}, \mathcal{K})$, where for all $\mathscr{k} \in \mathcal{K}$, $\mathfrak{Q}(\mathscr{k}) = \mathrm{U}$. Assume that $(\mathfrak{J}, \mathcal{K}) \mathrm{V}_\lambda \mathrm{U}_{\mathcal{K}} = (\mathfrak{J}, \mathcal{K}) \mathrm{V}_\lambda (\mathfrak{Q}, \mathcal{K}) = (\mathfrak{S}, \mathcal{K} \times \mathcal{K})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{K}$, $\mathfrak{S}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{Q}'(z) = \mathfrak{J}(\mathscr{k}) \cup \mathrm{U}' = \mathfrak{J}(\mathscr{k}) \cup \emptyset = \mathfrak{J}(\mathscr{k})$. Thus, $(\mathfrak{S}, \mathcal{K} \times \mathcal{K}) = (\mathfrak{J}, \mathcal{K} \mathrm{x} \mathcal{K})$. Let $\mathrm{U}_{\mathcal{K}} \mathrm{V}_\lambda (\mathfrak{J}, \mathcal{K}) =_{\mathrm{M}} (\mathfrak{P}, \mathcal{K} \mathrm{x} \mathcal{K})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{K}$, $\mathfrak{P}(\mathscr{k}, z) = \mathfrak{Q}(\mathscr{k}) \cup \mathfrak{J}'(z) = \mathrm{U} \cup \mathfrak{J}'(z) = \mathrm{U}$, thereby $(\mathfrak{P}, \mathcal{K} \mathrm{x} \mathcal{K}) = \mathrm{U}_{\mathcal{K} \times \mathcal{K}}$. □

Proposition 10. *Let* $(\mathfrak{J}, \mathcal{K})$ *be an SS over* U. *Then,* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_J (\mathfrak{J}, \mathcal{K}) V_\lambda (\mathfrak{J}, \mathcal{K})$. *That is,* V_λ*-product is not idempotent in* $S_E(U)$ *under J-equality.*

Proof 10. Let $(\mathfrak{J}, \mathcal{K}) \mathrm{V}_\lambda (\mathfrak{J}, \mathcal{K}) = (\mathfrak{S}, \mathcal{K} \times \mathcal{K})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{K}$, $\mathfrak{S}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{J}'(z)$. Since for all $\mathscr{k} \in \mathcal{K}$, there exists $(\mathscr{k}, z) \in \mathcal{K} \mathrm{x} \mathcal{K}$ such that $\mathfrak{J}(\mathscr{k}) \subseteq \mathfrak{S}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{J}'(z)$, $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{J}, \mathcal{K}) \mathrm{V}_\lambda (\mathfrak{J}, \mathcal{K})$ is obtained. □

Proposition 11. *Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{S}, \mathcal{Z})^r \tilde{\subseteq}_J (\mathfrak{J}, \mathcal{K}) V_\lambda (\mathfrak{S}, \mathcal{Z})$ *ve* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_J (\mathfrak{J}, \mathcal{K}) V_\lambda (\mathfrak{S}, \mathcal{Z})$.

Proof 11. Let $(\mathfrak{J}, \mathcal{K}) \mathrm{V}_\lambda (\mathfrak{S}, \mathcal{Z}) = (\mathfrak{Q}, \mathcal{K} \times \mathcal{Z})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{Q}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$. Since for all $z \in \mathcal{Z}$, there exists $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$ such that $\mathfrak{S}'(z) \subseteq \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$, $(\mathfrak{S}, \mathcal{Z})^{\mathrm{r}} \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{J}, \mathcal{K}) \mathrm{V}_\lambda (\mathfrak{S}, \mathcal{Z})$ is obtained. Similarly, since for all $\mathscr{k} \in \mathcal{K}$, there exists $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$ such that $\mathfrak{J}(\mathscr{k}) \subseteq \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$, $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{J}, \mathcal{K}) \mathrm{V}_\lambda (\mathfrak{S}, \mathcal{Z})$ is obtained. □

Proposition 12. *Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be SSs over* U. *Then,* $[(\mathfrak{J}, \mathcal{K}) V_\lambda (\mathfrak{S}, \mathcal{Z})]^r = (\mathfrak{J}, \mathcal{K})^r \wedge_\backslash (\mathfrak{S}, \mathcal{Z})^r$.

Proof 12. Let $(\mathfrak{J}, \mathcal{K}) \mathrm{V}_\lambda (\mathfrak{S}, \mathcal{Z}) = (\mathfrak{Q}, \mathcal{K} \times \mathcal{Z})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{Q}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$. Thus, $\mathfrak{Q}'(\mathscr{k}, z) = \mathfrak{J}'(\mathscr{k}) \cap \mathfrak{S}(z) = (\mathfrak{J})'(\mathscr{k}) \cap (\mathfrak{S}')'(z)$, implying that $(\mathfrak{Q}', \mathcal{K} \times \mathcal{Z}) = (\mathfrak{J}, \mathcal{K})^{\mathrm{r}} \wedge_\backslash (\mathfrak{S}, \mathcal{Z})^{\mathrm{r}}$. (For more about $\wedge_\backslash$-product, please see [11]. □

Proposition 13. *Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J}, \mathcal{K}) \wedge_\backslash (\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_F (\mathfrak{J}, \mathcal{K}) V_\lambda (\mathfrak{S}, \mathcal{Z})$.

Proof 13. Let $(\mathfrak{J}, \mathcal{K}) \wedge_\backslash (\mathfrak{S}, \mathcal{Z}) = (\mathfrak{E}, \mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J}, \mathcal{K}) \mathrm{V}_\lambda (\mathfrak{S}, \mathcal{Z}) = (\zeta, \mathcal{K} \times \mathcal{Z})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{E}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cap \mathfrak{S}'(z)$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\zeta(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$. Thus, for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{E}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cap \mathfrak{S}'(z) \subseteq \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z) = \zeta(\mathscr{k}, z)$. This completes the proof. □

Proposition 14. *Let* $(\mathfrak{J}, \mathcal{K})$, $(\mathfrak{S}, \mathcal{Z})$ *and* $(\mathfrak{Q}, \mathcal{C})$ *be SSs over* U. *If* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{Z})$, *then* $(\mathfrak{J}, \mathcal{K}) V_\lambda (\mathfrak{Q}, \mathcal{C}) \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{Z}) V_\lambda (\mathfrak{Q}, \mathcal{C})$.

Proof 14. Let $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{S}, \mathcal{Z})$. Then, $\mathcal{K} \subseteq \mathcal{Z}$ and for all $\mathscr{k} \in \mathcal{K}$, $\mathfrak{J}(\mathscr{k}) \subseteq \mathfrak{S}(\mathscr{k})$. Thus, $\mathcal{K} \times \mathcal{C} \subseteq \mathcal{Z} \times \mathcal{C}$ and for all $(\mathscr{k}, c) \in \mathcal{K} \times \mathcal{C}$, $\mathfrak{J}(\mathscr{k}) \cup \mathfrak{Q}'(c) \subseteq \mathfrak{S}(\mathscr{k}) \cup \mathfrak{Q}'(c)$, completing the proof. □

Proposition 15. *Let* $(\mathfrak{J}, \mathcal{K}), (\mathfrak{S}, \mathcal{Z}), (\mathfrak{Q}, \mathcal{C})$ *and* $(\mathfrak{G}, \mathcal{W})$ *be SSs over* U. *If* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{Z})$ *and* $(\mathfrak{Q}, \mathcal{C})^r \tilde{\subseteq}_F (\mathfrak{G}, \mathcal{W})^r$, *then* $(\mathfrak{J}, \mathcal{K}) V_\lambda (\mathfrak{Q}, \mathcal{C}) \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{Z}) V_\lambda (\mathfrak{G}, \mathcal{W})$.

Proof 15. Let $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{S}, \mathcal{Z})$ and $(\mathfrak{Q}, \mathcal{C})^{\mathrm{r}} \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{G}, \mathcal{W})^{\mathrm{r}}$. Thus, $\mathcal{K} \subseteq \mathcal{Z}$, $\mathcal{C} \subseteq \mathcal{W}$, for all $\mathscr{k} \in \mathcal{K}, \mathfrak{J}(\mathscr{k}) \subseteq \mathfrak{S}(\mathscr{k})$ and for all $c \in \mathcal{C}, \mathfrak{Q}'(c) \subseteq \mathfrak{G}'(c)$. Thus, $\mathcal{K} \times \mathcal{C} \subseteq \mathcal{Z} \times \mathcal{W}$, and for all $(\mathscr{k}, c) \in \mathcal{K} \times \mathcal{C}$, $\mathfrak{J}(\mathscr{k}) \cup \mathfrak{Q}'(c) \subseteq \mathfrak{S}(\mathscr{k}) \cup \mathfrak{G}'(c)$. This completes the proof. □

Proposition 16. *Let* $(\mathfrak{J}, \mathcal{K}), (\mathfrak{S}, \mathcal{K}), (\mathfrak{Q}, \mathcal{K})$ *and* $(\mathfrak{G}, \mathcal{K})$ *be SSs over* U. *If* $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{K})$ *and* $(\mathfrak{Q}, \mathcal{K}) \tilde{\subseteq}_F (\mathfrak{G}, \mathcal{K})$, *then* $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{G}, \mathcal{K}) \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{K})V_\lambda(\mathfrak{Q}, \mathcal{K})$.

Proof 16. Let $(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{K})$ and $(\mathfrak{Q}, \mathcal{K}) \tilde{\subseteq}_F (\mathfrak{G}, \mathcal{K})$. Thus, for all $\mathscr{k} \in \mathcal{K}$, $\mathfrak{J}(\mathscr{k}) \subseteq \mathfrak{S}(\mathscr{k})$ and for all $\ell \in \mathcal{K}$ için $\mathfrak{Q}(\ell) \subseteq \mathfrak{G}(\ell)$. Then, for all $(\mathscr{k}, \ell) \in \mathcal{K} \times \mathcal{K}$, $\mathfrak{J}(\mathscr{k}) \cup \mathfrak{G}'(\ell) \subseteq \mathfrak{S}(\mathscr{k}) \cup \mathfrak{Q}'(\ell)$, completing the proof. □

Proposition 17. *Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be SSs over* U. *Then,* $\emptyset_{\mathcal{K}\times\mathcal{Z}} \tilde{\subseteq}_F (\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z})$ *and* $\emptyset_{\mathcal{Z}\times\mathcal{K}} \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{Z})V_\lambda(\mathfrak{J}, \mathcal{K})$.

Proof 17. Let $\emptyset_{\mathcal{K}\times\mathcal{Z}} = (\mathcal{E}, \mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) = (\mathcal{S}, \mathcal{K} \times \mathcal{Z})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{E}(\mathscr{k}, z) = \emptyset$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{S}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$. Since $\mathcal{K} \times \mathcal{Z} \subseteq \mathcal{K} \times \mathcal{Z}$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{E}(\mathscr{k}, z) = \emptyset \subseteq \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z) = \mathcal{S}(\mathscr{k}, z)$, $\emptyset_{\mathcal{K}\times\mathcal{Z}} \tilde{\subseteq}_F (\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z})$ is obtained. Similarly, $\emptyset_{\mathcal{Z}\times\mathcal{K}} \tilde{\subseteq}_F (\mathfrak{S}, \mathcal{Z})V_\lambda(\mathfrak{J}, \mathcal{K})$. □

Proposition 18. *Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be SSs over* U. *Then,* $\emptyset_{\mathcal{K}} \tilde{\subseteq}_J (\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z})$, $\emptyset_{\mathcal{K}} \tilde{\subseteq}_J (\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z})$, *and* $\emptyset_E \tilde{\subseteq}_J (\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z})$.

Proof 18. Let $\emptyset_{\mathcal{K}} = (\mathcal{E}, \mathcal{K})$ and $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) = (\mathcal{S}, \mathcal{K} \times \mathcal{Z})$, where for all $\mathscr{k} \in \mathcal{K}$, $\mathcal{E}(\mathscr{k}) = \emptyset$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{S}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$. Since for all $\mathscr{k} \in \mathcal{K}$, there exists $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$ such that $\mathcal{E}(\mathscr{k}) = \emptyset \subseteq \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z) = \mathcal{S}(\mathscr{k}, z)$, $\emptyset_{\mathcal{K}} \tilde{\subseteq}_J (\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z})$ is obtained. Similarly, $\emptyset_{\mathcal{K}} \tilde{\subseteq}_J (\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z})$ and $\emptyset_E \tilde{\subseteq}_J (\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z})$ are obtained. □

Proposition 19. *Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_F U_{\mathcal{K}\times\mathcal{Z}}$ *and* $(\mathfrak{S}, \mathcal{Z})V_\lambda(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_F U_{\mathcal{Z}\times\mathcal{K}}$.

Proof 19. Let $U_{\mathcal{K}\times\mathcal{Z}} = (\mathfrak{Q}, \mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) = (\mathfrak{G}, \mathcal{K} \times \mathcal{Z})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{Q}(\mathscr{k}, z) = U$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{G}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$. Since $\mathcal{K} \times \mathcal{Z} \subseteq \mathcal{K} \times \mathcal{Z}$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{G}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z) \subseteq U = \mathfrak{Q}(\mathscr{k}, z)$, $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_F U_{\mathcal{K}\times\mathcal{Z}}$ is obtained. Similarly, $(\mathfrak{S}, \mathcal{Z})V_\lambda(\mathfrak{J}, \mathcal{K}) \tilde{\subseteq}_F U_{\mathcal{Z}\times\mathcal{K}}$. □

Proposition 20. *Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_J U_{\mathcal{K}}$ *and* $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_J U_{\mathcal{Z}}$.

Proof 20. Let $U_{\mathcal{K}} = (\mathcal{W}, \mathcal{K})$ and $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) = (\mathcal{X}, \mathcal{K} \times \mathcal{Z})$, where for all $\mathscr{k} \in \mathcal{K}$, $\mathcal{W}(\mathscr{k}) = U$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$ için, $\mathcal{X}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$. Since for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, there exists $\mathscr{k} \in \mathcal{K}$ such that $\mathcal{X}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z) \subseteq U = \mathcal{W}(\mathscr{k})$, $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_J U_{\mathcal{K}}$ is obtained. Similarly, $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) \tilde{\subseteq}_J U_{\mathcal{Z}}$. □

Proposition 21. *Let* $(\mathfrak{J}, \mathcal{K})$ *and* $(\mathfrak{S}, \mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) =_M \emptyset_{\mathcal{K}\times\mathcal{Z}}$ *if and only if* $(\mathfrak{J}, \mathcal{K}) =_M \emptyset_{\mathcal{K}}$ *and* $(\mathfrak{S}, \mathcal{Z}) =_M U_{\mathcal{Z}}$.

Proof 21. Let $\emptyset_{\mathcal{K}\times\mathcal{Z}} = (\mathcal{E}, \mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) = (\mathcal{X}, \mathcal{K} \times \mathcal{Z})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{E}(\mathscr{k}, z) = \emptyset$ and for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{X}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$. Let $(\mathcal{E}, \mathcal{K} \times \mathcal{Z}) = (\mathcal{X}, \mathcal{K} \times \mathcal{Z})$. Then, for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z) = \emptyset$. Thereby, for all $\mathscr{k} \in \mathcal{K}$, $\mathfrak{J}(\mathscr{k}) = \emptyset$ and for all $z \in \mathcal{Z}$, $\mathfrak{S}'(z) = \emptyset$. Therefore, for all $\mathscr{k} \in \mathcal{K}$, $\mathfrak{J}(\mathscr{k}) = \emptyset$ and for all $z \in \mathcal{Z}$, $\mathfrak{S}(z) = U$, implying that $(\mathfrak{J}, \mathcal{K}) =_M \emptyset_{\mathcal{K}}$ and $(\mathfrak{S}, \mathcal{Z}) =_M U_{\mathcal{Z}}$.

Conversely, let $(\mathfrak{J},\mathcal{K}) =_M \emptyset_{\mathcal{K}}$ and $(\mathfrak{S},\mathcal{Z}) =_M U_{\mathcal{Z}}$. Thus, for all $\mathscr{k} \in \mathcal{K}$, $\mathfrak{J}(\mathscr{k}) = \emptyset$ and for all $z \in \mathcal{Z}$, $\mathfrak{S}(z) = U$. Hence, for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathcal{X}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z) = \emptyset \cup \emptyset = \emptyset$, and so $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) =_M \emptyset_{\mathcal{K}\times\mathcal{Z}}$. □

Proposition 22. *Let* $(\mathfrak{J},\mathcal{K})$ *and* $(\mathfrak{S},\mathcal{Z})$ *be SSs over* U. *Then,* $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) =_M \emptyset_\emptyset$ *if and only if* $(\mathfrak{J},\mathcal{K}) =_M \emptyset_\emptyset$ *or* $(\mathfrak{S},\mathcal{Z}) =_M \emptyset_\emptyset$.

Proof 22. Let $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) =_M \emptyset_\emptyset$. Thereby, $\mathcal{K} \times \mathcal{Z} = \emptyset$, and so $\mathcal{K} = \emptyset$ or $\mathcal{Z} = \emptyset$. Since $\emptyset_\emptyset$ is the only SS with the empty PS, $(\mathfrak{J},\mathcal{K}) =_M \emptyset_\emptyset$ or $(\mathfrak{S},\mathcal{Z}) =_M \emptyset_\emptyset$.

Conversely, let $(\mathfrak{J},\mathcal{K}) = \emptyset_\emptyset$ or $(\mathfrak{S},\mathcal{Z}) = \emptyset_\emptyset$. Then, $\mathcal{K} = \emptyset$ or $\mathcal{Z} = \emptyset$. Since $\mathcal{K} \times \mathcal{Z} = \emptyset$ and $\emptyset_\emptyset$ is the only SS with empty PS, $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) =_M \emptyset_\emptyset$. □

4. Distributions of soft lambda-product over certaın types of soft set's operatıons

In this section, we investigate the distributions of soft lambda-product over restricted, extended, soft binary piecewise intersection and union operations, AND-product and OR-product.

Theorem 1. *Let* $(\mathfrak{J},\mathcal{K}), (\mathfrak{S},\mathcal{Z})$ *and* $(\mathfrak{Q},\mathcal{C})$ *be SSs over* U. *Then, we have the following distributions of soft lambda-product over restricted intersection and union operations:*

i) $(\mathfrak{J},\mathcal{K})V_\lambda[(\mathfrak{S},\mathcal{Z}) \cup_R (\mathfrak{Q},\mathcal{C})] =_M [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})] \cap_R [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\mathcal{C})]$,

ii) $(\mathfrak{J},\mathcal{K})V_\lambda[(\mathfrak{S},\mathcal{Z}) \cap_R (\mathfrak{Q},\mathcal{C})] =_M [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})] \cup_R [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\mathcal{C})]$,

iii) $[(\mathfrak{S},\mathcal{Z}) \cap_R (\mathfrak{Q},\mathcal{C})]V_\lambda(\mathfrak{J},\mathcal{K}) =_M [(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})] \cap_R [(\mathfrak{Q},\mathcal{C})V_\lambda(\mathfrak{J},\mathcal{K})]$,

iv) $[(\mathfrak{S},\mathcal{Z}) \cup_R (\mathfrak{Q},\mathcal{C})]V_\lambda(\mathfrak{J},\mathcal{K}) =_M [(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})] \cup_R [(\mathfrak{Q},\mathcal{C})V_\lambda(\mathfrak{J},\mathcal{K})]$,

Proof 23. (i) The PS of the left-hand side (LHS) is $\mathcal{K}\text{x}(\mathcal{Z} \cap \mathcal{C})$, and the PS of the right-hand side (RHS) is $(\mathcal{K}\text{x}\mathcal{Z}) \cap (\mathcal{K}\text{x}\mathcal{C})$. Since $\mathcal{K}\text{x}(\mathcal{Z} \cap \mathcal{C}) = (\mathcal{K}\text{x}\mathcal{Z}) \cap (\mathcal{K}\text{x}\mathcal{C})$, the first condition of the M-equality is satisfied. Let $(\mathfrak{S},\mathcal{Z}) \cup_R (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z} \cap \mathcal{C})$, where for all $\mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}$, $\mathfrak{E}(\mathfrak{z}) = \mathfrak{S}(\mathfrak{z}) \cup \mathfrak{Q}(\mathfrak{z})$. Let $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{E},\mathcal{Z} \cap \mathcal{C}) = \big(\wp, \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C})\big)$, where for all $(\mathscr{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C})$, $\wp(\mathscr{k},\mathfrak{z}) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{E}'(\mathfrak{z})$. Thus,

$$\wp(\mathscr{k},\mathfrak{z}) = \mathfrak{J}(\mathscr{k}) \cup [\mathfrak{S}(\mathfrak{z}) \cup \mathfrak{Q}(\mathfrak{z})]' = \mathfrak{J}(\mathscr{k}) \cup [\mathfrak{S}'(\mathfrak{z}) \cap \mathfrak{Q}'(\mathfrak{z})].$$

Let $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) = (\mathfrak{M},\mathcal{K} \times \mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\mathcal{C}) = (\mathfrak{P},\mathcal{K} \times \mathcal{C})$, where for all $(\mathscr{k}, z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{M}(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$ and for all $(\mathscr{k}, c) \in \mathcal{K} \times \mathcal{C}$, $\mathfrak{P}(\mathscr{k}, c) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{Q}'(c)$. Suppose that $(\mathfrak{M},\mathcal{K} \times \mathcal{Z}) \cap_R (\mathfrak{P},\mathcal{K} \times \mathcal{C}) = \big(\mathfrak{R},(\mathcal{K} \times \mathcal{Z}) \cap (\mathcal{K} \times \mathcal{C})\big)$, where for all $(\mathscr{k},\mathfrak{z}) \in (\mathcal{K} \times \mathcal{Z}) \cap (\mathcal{K} \times \mathcal{C}) = \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C})$,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z}) = \mathfrak{M}(\mathscr{k},\mathfrak{z}) \cap \mathfrak{P}(\mathscr{k},\mathfrak{z}) = [\mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(\mathfrak{z})] \cap [\mathfrak{J}(\mathscr{k}) \cup \mathfrak{Q}'(\mathfrak{z})].$$

Thus, $(\mathfrak{J},\mathcal{K})V_\lambda[(\mathfrak{S},\mathcal{Z}) \cup_R (\mathfrak{Q},\mathcal{C})] =_M [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})] \cap_R [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\mathcal{C})]$.

Here, if $\mathcal{Z} \cap \mathcal{C} = \emptyset$, then $\mathcal{K}\text{x}(\mathcal{Z} \cap \mathcal{C}) = (\mathcal{K}\text{x}\mathcal{Z}) \cap (\mathcal{K}\text{x}\mathcal{C}) = \emptyset$. Since the only SS with an empty PS is $\emptyset_\emptyset$, then both sides are $\emptyset_\emptyset$. Since $(\mathcal{K}\text{x}\mathcal{Z}) \cap (\mathcal{K}\text{x}\mathcal{C}) = \mathcal{K}\text{x}(\mathcal{Z} \cap \mathcal{C})$, if $(\mathcal{K}\text{x}\mathcal{Z}) \cap (\mathcal{K}\text{x}\mathcal{C})=\emptyset$, then $\mathcal{K}=\emptyset$ or $\mathcal{Z} \cap \mathcal{C} = \emptyset$. By assumption, $\mathcal{K} \neq \emptyset$. Thus, $(\mathcal{K}\text{x}\mathcal{Z}) \cap (\mathcal{K}\text{x}\mathcal{C}) = \emptyset$ implies that $\mathcal{Z} \cap \mathcal{C} = \emptyset$. Therefore, under this condition, both sides are again $\emptyset_\emptyset$.

(iii) The PS of the LHS is $(\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$, and the PS of the RHS is $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K})$, and since $(\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K} = (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K})$, the first condition of M-equality is satisfied. Let $(\mathfrak{S},\mathcal{Z}) \cap_R (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z} \cap \mathcal{C})$, where for all $\mathfrak{z} \in \mathcal{Z} \cap \mathcal{C}$, $\mathfrak{E}(\mathfrak{z}) = \mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z})$. Let $(\mathfrak{E},\mathcal{Z} \cap \mathcal{C})V_\lambda(\mathfrak{J},\mathcal{K}) = (\wp,(\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}))$, where for all

$(\mathfrak{z},\mathcal{k}) \in (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$, $\wp(\mathfrak{z},\mathcal{k}) = \mathfrak{E}(\mathfrak{z}) \cup \mathfrak{J}'(\mathcal{k})$. Thus,

$$\wp(\mathfrak{z},\mathcal{k}) = [\mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z})] \cup \mathfrak{J}'(\mathcal{k}).$$

Assume that $(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K}) = (\mathfrak{M},\mathcal{Z} \times \mathcal{K})$ and $(\mathfrak{Q},\mathcal{C})V_\lambda(\mathfrak{J},\mathcal{K}) = (\mathfrak{P},\mathcal{C} \times \mathcal{K})$, where for all $(z,\mathcal{k}) \in \mathcal{Z} \times \mathcal{K}$, $\mathfrak{M}(z,\mathcal{k}) = \mathfrak{S}(\mathfrak{z}) \cup \mathfrak{J}'(\mathcal{k})$ and for all $(c,\mathcal{k}) \in \mathcal{C} \times \mathcal{K}$, $\mathfrak{P}(c,\mathcal{k}) = \mathfrak{Q}(c) \cup \mathfrak{J}'(\mathcal{k})$. Let $(\mathfrak{M},\mathcal{Z} \times \mathcal{K}) \cap_R (\mathfrak{P},\mathcal{C} \times \mathcal{K}) = \big(\mathfrak{R},(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K})\big)$, where for all $(\mathfrak{z},\mathcal{k}) \in (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$,

$$\mathfrak{R}(\mathfrak{z},\mathcal{k}) = \mathfrak{M}(\mathfrak{z},\mathcal{k}) \cap \mathfrak{P}(\mathfrak{z},\mathcal{k}) = [\mathfrak{S}(\mathfrak{z}) \cup \mathfrak{J}'(\mathcal{k})] \cap [\mathfrak{Q}(\mathfrak{z}) \cup \mathfrak{J}'(\mathcal{k})].$$

Thus, $[(\mathfrak{S},\mathcal{Z}) \cap_R (\mathfrak{Q},\mathcal{C})]V_\lambda(\mathfrak{J},\mathcal{K}) =_M ([\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})] \cap_R [(\mathfrak{Q},\mathcal{C})V_\lambda(\mathfrak{J},\mathcal{K})]$.

Here if $\mathcal{Z} \cap \mathcal{C} = \emptyset$, then $(\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K} = (\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K})=\emptyset$. Since the only SS with the empty parameter set is $\emptyset_\emptyset$, then both sides of the equality are $\emptyset_\emptyset$. Similarly since $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = (\mathcal{Z} \cap \mathcal{C}) \times \mathcal{K}$, if $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = \emptyset$, then $\mathcal{Z} \cap \mathcal{C} = \emptyset$ or $\mathcal{K} = \emptyset$. By assumption $\mathcal{K} \neq \emptyset$. Hence, $(\mathcal{Z} \times \mathcal{K}) \cap (\mathcal{C} \times \mathcal{K}) = \emptyset$ implies that $\mathcal{Z} \cap \mathcal{C} = \emptyset$. Thus, under this condition, both sides of the equality are again $\emptyset_\emptyset$. □

Note 1. *The restricted SS operation cannot distribute over soft lambda-product as the intersection does not distribute over cartesian product and it is compulsory for two SSs to be M-equal that their PS should be the same.*

Theorem 2. *Let $(\mathfrak{J},\mathcal{K})$, $(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ be SSs over U. Then, we have the following distributions of soft lambda-product over extended intersection and union operations:*

i) $(\mathfrak{J},\mathcal{K})V_\lambda[(\mathfrak{S},\mathcal{Z}) \cap_\varepsilon (\mathfrak{Q},\mathcal{C})] =_M [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})] \cup_\varepsilon [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\mathcal{C})]$,

ii) $(\mathfrak{J},\mathcal{K})V_\lambda[(\mathfrak{S},\mathcal{Z}) \cup_\varepsilon (\mathfrak{Q},\mathcal{C})] =_M [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})] \cap_\varepsilon [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\mathcal{C})]$,

iii) $[(\mathfrak{S},\mathcal{Z}) \cup_\varepsilon (\mathfrak{Q},\mathcal{C})]V_\lambda(\mathfrak{J},\mathcal{K}) =_M [(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})] \cup_\varepsilon [(\mathfrak{Q},\mathcal{C})V_\lambda(\mathfrak{J},\mathcal{K})]$,

iii) $[(\mathfrak{S},\mathcal{Z}) \cap_\varepsilon (\mathfrak{Q},\mathcal{C})]V_\lambda(\mathfrak{J},\mathcal{K}) =_M [(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})] \cap_\varepsilon [(\mathfrak{Q},\mathcal{C})V_\lambda(\mathfrak{J},\mathcal{K})]$.

Proof 24. (i) The PS of the LHS is $\mathcal{K}\mathrm{x}(\mathcal{Z} \cup \mathcal{C})$, and the PS of the RHS is $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cup (\mathcal{K}\mathrm{x}\mathcal{C})$. Since $\mathcal{K}\mathrm{x}(\mathcal{Z} \cup \mathcal{C}) = (\mathcal{K}\mathrm{x}\mathcal{Z}) \cup (\mathcal{K}\mathrm{x}\mathcal{C})$, the first condition of the M-equality is satisfied. As $\mathcal{K} \neq \emptyset$, $\mathcal{Z} \neq \emptyset$, and $\mathcal{C} \neq \emptyset$, $\mathcal{K}\mathrm{x}(\mathcal{Z} \cup \mathcal{C}) \neq \emptyset$ and $(\mathcal{K}\mathrm{x}\mathcal{Z}) \cup (\mathcal{K}\mathrm{x}\mathcal{C}) \neq \emptyset$. No side can therefore be equivalent to an empty SS. Let $(\mathfrak{S},\mathcal{Z}) \cap_\varepsilon (\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z} \cup \mathcal{C})$, where for all $\mathfrak{z} \in \mathcal{Z} \cup \mathcal{C}$,

$$\mathfrak{E}(\mathfrak{z}) = \begin{cases} \mathfrak{S}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} - \mathcal{C} \\ \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{C} - \mathcal{Z} \\ \mathfrak{S}(\mathfrak{z}) \cap \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z} \in \mathcal{Z} \cap \mathcal{C} \end{cases}.$$

Let $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{E},\mathcal{Z} \cup \mathcal{C}) = \big(\mathbb{Q},\mathcal{K} \times (\mathcal{Z} \cup \mathcal{C})\big)$, where for all $(\mathcal{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cup \mathcal{C})$, $\mathbb{Q}(\mathcal{k},\mathfrak{z}) = \mathfrak{J}(\mathcal{k}) \cup \mathfrak{E}'(\mathfrak{z})$. Thus, for all $(\mathcal{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cup \mathcal{C})$,

$$\mathbb{Q}(\mathcal{k},\mathfrak{z}) = \begin{cases} \mathfrak{J}(\mathcal{k}) \cup \mathfrak{S}'(\mathfrak{z}), & (\mathcal{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} - \mathcal{C}) \\ \mathfrak{J}(\mathcal{k}) \cup \mathfrak{Q}'(\mathfrak{z}), & (\mathcal{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{C} - \mathcal{Z}) \\ \mathfrak{J}(\mathcal{k}) \cup [\mathfrak{S}'(\mathfrak{z}) \cup \mathfrak{Q}'(\mathfrak{z})], & (\mathcal{k},\mathfrak{z}) \in \mathcal{K} \times (\mathcal{Z} \cap \mathcal{C}) \end{cases}.$$

Let $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) = \big((\mathfrak{M},\mathcal{K} \times \mathcal{Z})\big)$ and $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\mathcal{C}) = (\mathfrak{P},\mathcal{K} \times \mathcal{C})$, where for all $(\mathcal{k},z) \in \mathcal{K} \times \mathcal{Z}$, $\mathfrak{M}(\mathcal{k},z) = \mathfrak{J}(\mathcal{k}) \cup \mathfrak{S}'(z)$ and for all $(\mathcal{k},c) \in \mathcal{K} \times \mathcal{C}$, $\mathfrak{P}(\mathcal{k},c) = \mathfrak{J}(\mathcal{k}) \cup \mathfrak{Q}'(c)$. Assume that $(\mathfrak{M},\mathcal{K} \times \mathcal{Z}) \cup_\varepsilon (\mathfrak{P},\mathcal{K} \times \mathcal{C}) = \big(\mathfrak{R},(\mathcal{K} \times \mathcal{Z}) \cup (\mathcal{K} \times \mathcal{C})\big)$, where for all $(\mathcal{k},\mathfrak{z}) \in (\mathcal{K} \times \mathcal{Z}) \cup (\mathcal{K} \times \mathcal{C}) = \mathcal{K} \times (\mathcal{Z} \cup \mathcal{C})$,

$$\mathfrak{R}(\ell,\mathfrak{z})=\begin{cases}\mathfrak{M}(\ell,\mathfrak{z}), & (\ell,\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})-(\mathcal{K}\times\mathcal{C})=\mathcal{K}\times(\mathcal{Z}-\mathcal{C})\\ \mathfrak{P}(\ell,\mathfrak{z}), & (\ell,\mathfrak{z})\in(\mathcal{K}\times\mathcal{C})-(\mathcal{K}\times\mathcal{Z})=\mathcal{K}\times(\mathcal{C}-\mathcal{Z})\\ \mathfrak{M}(\ell,\mathfrak{z})\cup\mathfrak{P}(\ell,\mathfrak{z}), & (\ell,\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})\cap(\mathcal{K}\times\mathcal{C})=\mathcal{K}\times(\mathcal{Z}\cap\mathcal{C})\end{cases}.$$

Thus,

$$\mathfrak{R}(\ell,\mathfrak{z})=\begin{cases}\mathfrak{J}(\ell)\cup\mathfrak{S}'(\mathfrak{z}), & (\ell,\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})-(\mathcal{K}\times\mathcal{C})=\mathcal{K}\times(\mathcal{Z}-\mathcal{C})\\ \mathfrak{J}(\ell)\cup\mathfrak{Q}'(\mathfrak{z}), & (\ell,\mathfrak{z})\in(\mathcal{K}\times\mathcal{C})-(\mathcal{K}\times\mathcal{Z})=\mathcal{K}\times(\mathcal{C}-\mathcal{Z})\\ [\mathfrak{J}(\ell)\cup\mathfrak{S}'(\mathfrak{z})]\cup[\mathfrak{J}(\ell)\cup\mathfrak{Q}'(\mathfrak{z})], & (\ell,\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})\cap(\mathcal{K}\times\mathcal{C})=\mathcal{K}\times(\mathcal{Z}\cap\mathcal{C})\end{cases}.$$

Hence, $(\mathfrak{J},\mathcal{K})\mathrm{V}_\lambda[(\mathfrak{S},\mathcal{Z})\cap_\varepsilon(\mathfrak{Q},\mathcal{C})]=_\mathrm{M}[(\mathfrak{J},\mathcal{K})\mathrm{V}_\lambda(\mathfrak{S},\mathcal{Z})]\cup_\varepsilon[(\mathfrak{J},\mathcal{K})\mathrm{V}_\lambda(\mathfrak{Q},\mathcal{C})]$

(iii) The PS of the LHS is $(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K}$, and the PS of the RHS is $(\mathcal{Z}\times\mathcal{K})\cup(\mathcal{C}\times\mathcal{K})$, and since $(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K}=(\mathcal{Z}\times\mathcal{K})\cup(\mathcal{C}\times\mathcal{K})$ the first condition of M-equality is satisfied. By assumption, $\mathcal{K}\neq\emptyset$, $\mathcal{Z}\neq\emptyset$, and $\mathcal{C}\neq\emptyset$. Thus, $(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K}\neq\emptyset$ and $(\mathcal{Z}\times\mathcal{K})\cup(\mathcal{C}\times\mathcal{K})\neq\emptyset$. No side can therefore be equivalent to an empty SS. Let $(\mathfrak{S},\mathcal{Z})\cup_\varepsilon(\mathfrak{Q},\mathcal{C})=(\mathfrak{E},\mathcal{Z}\cup\mathcal{C})$, where for all $\mathfrak{z}\in\mathcal{Z}\cup\mathcal{C}$,

$$\mathfrak{E}(\mathfrak{z})=\begin{cases}\mathfrak{S}(\mathfrak{z}), & \mathfrak{z}\in\mathcal{Z}-\mathcal{C}\\ \mathfrak{Q}(\mathfrak{z}), & \mathfrak{z}\in\mathcal{C}-\mathcal{Z}\\ \mathfrak{S}(\mathfrak{z})\cup\mathfrak{Q}(\mathfrak{z}), & \mathfrak{z}\in\mathcal{Z}\cap\mathcal{C}\end{cases}.$$

Let $(\mathfrak{E},\mathcal{Z}\cup\mathcal{C})\mathrm{V}_\lambda(\mathfrak{J},\mathcal{K})=(\wp,(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K})$, where for all $(\mathfrak{z},\ell)\in(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K}$, $\wp(\mathfrak{z},\ell)=\mathfrak{E}(\mathfrak{z})\cup\mathfrak{J}'(\ell)$,

$$\wp(\ell,\mathfrak{z})=\begin{cases}\mathfrak{S}(\mathfrak{z})\cup\mathfrak{J}'(\ell), & (\ell,\mathfrak{z})\in(\mathcal{Z}-\mathcal{C})\times\mathcal{K}\\ \mathfrak{Q}(\mathfrak{z})\cup\mathfrak{J}'(\ell), & (\ell,\mathfrak{z})\in(\mathcal{C}-\mathcal{Z})\times\mathcal{K}\\ [\mathfrak{S}(\mathfrak{z})\cup\mathfrak{Q}(\mathfrak{z})]\cup\mathfrak{J}'(\ell), & (\ell,\mathfrak{z})\in(\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}\end{cases}.$$

Let $(\mathfrak{S},\mathcal{Z})\mathrm{V}_\lambda(\mathfrak{J},\mathcal{K})=(\mathfrak{M},\mathcal{Z}\times\mathcal{K})$ and $(\mathfrak{Q},\mathcal{C})\mathrm{V}_\lambda(\mathfrak{J},\mathcal{K})=(\mathfrak{P},\mathcal{C}\times\mathcal{K})$, where for all $(z,\ell)\in\mathcal{Z}\times\mathcal{K}$, $\mathfrak{M}(z,\ell)=\mathfrak{S}(\mathfrak{z})\cup\mathfrak{J}'(\ell)$ and for all $(c,\ell)\in\mathcal{C}\times\mathcal{K}$, $\mathfrak{P}(c,\ell)=\mathfrak{Q}(c)\cup\mathfrak{J}'(\ell)$. Assume that $(\mathfrak{M},\mathcal{Z}\times\mathcal{K})\cup_\varepsilon(\mathfrak{P},\mathcal{C}\times\mathcal{K})=\big(\mathfrak{R},(\mathcal{Z}\times\mathcal{K})\cup(\mathcal{C}\times\mathcal{K})\big)$, where for all $(\mathfrak{z},\ell)\in(\mathcal{Z}\times\mathcal{K})\cup(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}\cup\mathcal{C})\times\mathcal{K}$,

$$\mathfrak{R}(\ell,\mathfrak{z})=\begin{cases}\mathfrak{M}(\mathfrak{z},\ell), & (\ell,\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})-(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}-\mathcal{C})\times\mathcal{K}\\ \mathfrak{P}(\mathfrak{z},\ell), & (\ell,\mathfrak{z})\in(\mathcal{C}\times\mathcal{K})-(\mathcal{Z}\times\mathcal{K})=(\mathcal{C}-\mathcal{Z})\times\mathcal{K}\\ \mathfrak{M}(\mathfrak{z},\ell)\cup\mathfrak{P}(\mathfrak{z},\ell), & (\ell,\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})\cap(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}\end{cases}.$$

Thus,

$$\mathfrak{R}(\ell,\mathfrak{z})=\begin{cases}\mathfrak{S}(\mathfrak{z})\cup\mathfrak{J}'(\ell), & (\ell,\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})-(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}-\mathcal{C})\times\mathcal{K}\\ \mathfrak{Q}(\mathfrak{z})\cup\mathfrak{J}'(\ell), & (\ell,\mathfrak{z})\in(\mathcal{C}\times\mathcal{K})-(\mathcal{Z}\times\mathcal{K})=(\mathcal{C}-\mathcal{Z})\times\mathcal{K}\\ [\mathfrak{S}(\mathfrak{z})\cup\mathfrak{J}'(\ell)]\cup[\mathfrak{Q}(\mathfrak{z})\cup\mathfrak{J}'(\ell)], & (\ell,\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})\cap(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}\end{cases}.$$

Hence, $[(\mathfrak{S},\mathcal{Z})\cup_\varepsilon(\mathfrak{Q},\mathcal{C})]\mathrm{V}_\lambda(\mathfrak{J},\mathcal{K})=_\mathrm{M}[(\mathfrak{S},\mathcal{Z})\mathrm{V}_\lambda(\mathfrak{J},\mathcal{K})]\cup_\varepsilon[(\mathfrak{Q},\mathcal{C})\mathrm{V}_\lambda(\mathfrak{J},\mathcal{K})]$.
□

Note 2. *The extended SS operation cannot distribute over soft lambda-product as the union operation does not distribute over cartesian product and it is compulsory for two SSs to be M-equal that their PS should be the same.*

Theorem 3. *Let $(\mathfrak{J},\mathcal{K})$, $(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ be SSs over U. Then, we have the following distributions of soft lambda-product over soft binary piecewise intersection and union operations:*

i) $(\mathfrak{J},\mathcal{K})\mathrm{V}_\lambda[(\mathfrak{S},\mathcal{Z})\,\tilde{\cap}\,(\mathfrak{Q},\mathcal{C})]=_\mathrm{M}[(\mathfrak{J},\mathcal{K})\mathrm{V}_\lambda(\mathfrak{S},\mathcal{Z})]\,\tilde{\cup}\,[(\mathfrak{J},\mathcal{K})\mathrm{V}_\lambda(\mathfrak{Q},\mathcal{C})]$,

ii) $(\mathfrak{J},\mathcal{K})V_\lambda[(\mathfrak{S},\mathcal{Z})\,\tilde{\cup}\,(\mathfrak{Q},\mathcal{C})] =_M [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})]\,\tilde{\cap}\,[(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\mathcal{C})]$,

iii) $[(\mathfrak{S},\mathcal{Z})\,\tilde{\cup}\,(\mathfrak{Q},\mathcal{C})]V_\lambda(\mathfrak{J},\mathcal{K}) =_M [(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})]\,\tilde{\cup}\,[(\mathfrak{Q},\mathcal{C})V_\lambda(\mathfrak{J},\mathcal{K})]$,

iv) $[(\mathfrak{S},\mathcal{Z})\,\tilde{\cap}\,(\mathfrak{Q},\mathcal{C})]V_\lambda(\mathfrak{J},\mathcal{K}) =_M [(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K})]\,\tilde{\cap}\,[(\mathfrak{Q},\mathcal{C})V_\lambda(\mathfrak{J},\mathcal{K})]$.

Proof 25. (i) Since the PS of the SSs of both sides are $\mathcal{K}\text{x}\mathcal{Z}$, the first condition of the M-equality is satisfied. Moreover since $\mathcal{K}\neq\emptyset$ and $\mathcal{Z}\neq\emptyset$ by assumption, $\mathcal{K}\text{x}\mathcal{Z}\neq\emptyset$. No side can therefore be equivalent to an empty SS. Let $(\mathfrak{S},\mathcal{Z})\,\tilde{\cap}\,(\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z})$, where for all $\mathfrak{z}\in\mathcal{Z}$,

$$\mathfrak{E}(\mathfrak{z}) = \begin{cases} \mathfrak{S}(\mathfrak{z}), & \mathfrak{z}\in\mathcal{Z}-\mathcal{C} \\ \mathfrak{S}(\mathfrak{z})\cap\mathfrak{Q}(\mathfrak{z}), & \mathfrak{z}\in\mathcal{Z}\cap\mathcal{C}. \end{cases}$$

Let $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{E},\mathcal{Z}) = (\mathbb{Q},\mathcal{K}\times\mathcal{Z})$, where for all $(\mathcal{k},\mathfrak{z})\in\mathcal{K}\times\mathcal{Z}$, $\mathbb{Q}(\mathcal{k},\mathfrak{z}) = \mathfrak{J}(\mathcal{k})\cup\mathfrak{E}'(\mathfrak{z})$. Thus,

$$\mathbb{Q}(\mathcal{k},\mathfrak{z}) = \begin{cases} \mathfrak{J}(\mathcal{k})\cup\mathfrak{S}'(\mathfrak{z}), & (\mathcal{k},\mathfrak{z})\in\mathcal{K}\times(\mathcal{Z}-\mathcal{C}) \\ \mathfrak{J}(\mathcal{k})\cup[\mathfrak{S}'(\mathfrak{z})\cup\mathfrak{Q}'(\mathfrak{z})], & (\mathcal{k},\mathfrak{z})\in\mathcal{K}\times\mathcal{Z}\cap\mathcal{C}. \end{cases}$$

Let $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z}) = (\mathfrak{M},\mathcal{K}\times\mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\mathcal{C}) = (\mathfrak{P},\mathcal{K}\times\mathcal{C})$, where for all $(\mathcal{k},z)\in\mathcal{K}\times\mathcal{Z}$, $\mathfrak{M}(\mathcal{k},z) = \mathfrak{J}(\mathcal{k})\cup\mathfrak{S}'(\mathfrak{z})$ and for all $(\mathcal{k},c)\in\mathcal{K}\times\mathcal{C}$, $\mathfrak{P}(\mathcal{k},c) = \mathfrak{J}(\mathcal{k})\cup\mathfrak{Q}'(c)$. Assume that $(\mathfrak{M},\mathcal{K}\times\mathcal{Z})\,\tilde{\cup}\,(\mathfrak{P},\mathcal{K}\times\mathcal{C}) = \big(\mathfrak{R},(\mathcal{K}\times\mathcal{Z})\big)$, where for all $(\mathcal{k},\mathfrak{z})\in\mathcal{K}\times\mathcal{Z}$,

$$\mathfrak{R}(\mathcal{k},\mathfrak{z}) = \begin{cases} \mathfrak{M}(\mathcal{k},\mathfrak{z}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})-(\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}-\mathcal{C}) \\ \mathfrak{M}(\mathcal{k},\mathfrak{z})\cup\mathfrak{P}(\mathcal{k},\mathfrak{z}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})\cap(\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}\cap\mathcal{C}). \end{cases}$$

Therefore,

$$\mathfrak{R}(\mathcal{k},\mathfrak{z}) = \begin{cases} \mathfrak{J}(\mathcal{k})\cup\mathfrak{S}'(\mathfrak{z}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})-(\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}-\mathcal{C}) \\ [\mathfrak{J}(\mathcal{k})\cup\mathfrak{S}'(\mathfrak{z})]\cup[\mathfrak{J}(\mathcal{k})\cup\mathfrak{Q}'(\mathfrak{z})], & (\mathcal{k},\mathfrak{z})\in(\mathcal{K}\times\mathcal{Z})\cap(\mathcal{K}\times\mathcal{C}) = \mathcal{K}\times(\mathcal{Z}\cap\mathcal{C}) \end{cases}.$$

Thus, $(\mathfrak{J},\mathcal{K})V_\lambda[(\mathfrak{S},\mathcal{Z})\,\tilde{\cap}\,(\mathfrak{Q},\mathcal{C})] =_M [(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{S},\mathcal{Z})]\,\tilde{\cup}\,[(\mathfrak{J},\mathcal{K})V_\lambda(\mathfrak{Q},\mathcal{C})]$.

Since $\mathcal{K}\neq\mathcal{K}\text{x}\mathcal{K}$, the soft binary piecewise operations do not distribute over soft lambda-product operations.

(iii) Since the PS of the SSs of both sides are $\mathcal{Z}\times\mathcal{K}$, the first condition of the M-equality is satisfied. Moreover since $\mathcal{Z}\neq\emptyset$ and $\mathcal{K}\neq\emptyset$ by assumption, $\mathcal{Z}\times\mathcal{K}\neq\emptyset$. No side can therefore be equivalent to an empty SS. Let $(\mathfrak{S},\mathcal{Z})\,\tilde{\cup}\,(\mathfrak{Q},\mathcal{C}) = (\mathfrak{E},\mathcal{Z})$, where for all $\mathfrak{z}\in\mathcal{Z}$,

$$\mathfrak{E}(\mathfrak{z}) = \begin{cases} \mathfrak{S}(\mathfrak{z}), & \mathfrak{z}\in\mathcal{Z}-\mathcal{C} \\ \mathfrak{S}(\mathfrak{z})\cup\mathfrak{Q}(\mathfrak{z}), & \mathfrak{z}\in\mathcal{Z}\cap\mathcal{C}. \end{cases}$$

Let $(\mathfrak{E},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K}) = (\wp,\mathcal{Z}\times\mathcal{K})$, where for all $(\mathfrak{z},\mathcal{k})\in\mathcal{Z}\times\mathcal{K}$, $\wp(\mathfrak{z},\mathcal{k}) = \mathfrak{E}(\mathfrak{z})\cup\mathfrak{J}'(\mathcal{k})$. Thus,

$$\wp(\mathcal{k},\mathfrak{z}) = \begin{cases} \mathfrak{S}(\mathfrak{z})\cup\mathfrak{J}'(\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{Z}-\mathcal{C})\times\mathcal{K} \\ [\mathfrak{S}(\mathfrak{z})\cup\mathfrak{Q}(\mathfrak{z})]\cup\mathfrak{J}'(\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{Z}\cap\mathcal{C})\times\mathcal{K} \end{cases}.$$

Let $(\mathfrak{S},\mathcal{Z})V_\lambda(\mathfrak{J},\mathcal{K}) = (\mathfrak{M},\mathcal{Z}\times\mathcal{K})$ and $(\mathfrak{Q},\mathcal{C})V_\lambda(\mathfrak{J},\mathcal{K}) = (\mathfrak{P},\mathcal{C}\times\mathcal{K})$, where for all $(z,\mathcal{k})\in\mathcal{Z}\times\mathcal{K}$, $\mathfrak{M}(z,\mathcal{k}) = \mathfrak{S}(z)\cup\mathfrak{J}'(\mathcal{k})$ and for all $(c,\mathcal{k})\in\mathcal{C}\times\mathcal{K}$, $\mathfrak{P}(c,\mathcal{k}) = \mathfrak{Q}(c)\cup\mathfrak{J}'(\mathcal{k})$. Assume that $(\mathfrak{M},\mathcal{Z}\times\mathcal{K})\,\tilde{\cup}\,(\mathfrak{P},\mathcal{C}\times\mathcal{K}) = \big(\mathfrak{R},(\mathcal{Z}\times\mathcal{K})\big)$, where for all $(\mathfrak{z},\mathcal{k})\in(\mathcal{Z}\times\mathcal{K})$,

$$\mathfrak{R}(\mathcal{k},\mathfrak{z}) = \begin{cases} \mathfrak{M}(\mathfrak{z},\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})-(\mathcal{C}\times\mathcal{K}) = (\mathcal{Z}-\mathcal{C})\times\mathcal{K} \\ \mathfrak{M}(\mathfrak{z},\mathcal{k})\cup\mathfrak{P}(\mathfrak{z},\mathcal{k}), & (\mathcal{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})\cap(\mathcal{C}\times\mathcal{K}) = (\mathcal{Z}\cap\mathcal{C})\times\mathcal{K} \end{cases}.$$

Thus,

$$\mathfrak{R}(\mathscr{k},\mathfrak{z})=\begin{cases}\mathfrak{S}(\mathfrak{z})\cup\mathfrak{J}'(\mathscr{k}), & (\mathscr{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})-(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}-\mathcal{C})\times\mathcal{K}\\ [\mathfrak{S}(\mathfrak{z})\cup\mathfrak{J}'(\mathscr{k})]\cup[\mathfrak{Q}(\mathfrak{z})\cup\mathfrak{J}'(\mathscr{k})], & (\mathscr{k},\mathfrak{z})\in(\mathcal{Z}\times\mathcal{K})\cap(\mathcal{C}\times\mathcal{K})=(\mathcal{Z}\cap\mathcal{C})\times\mathcal{K}\end{cases}.$$

Hence, $[(\mathfrak{S},\mathcal{Z})\,\tilde{\cup}\,(\mathfrak{Q},\mathcal{C})]\vee_\lambda(\mathfrak{J},\mathcal{K})=_M[(\mathfrak{S},\mathcal{Z})\vee_\lambda(\mathfrak{J},\mathcal{K})]\,\tilde{\cup}\,[(\mathfrak{Q},\mathcal{C})\vee_\lambda(\mathfrak{J},\mathcal{K})]$. □

Proposition 23. *Let $(\mathfrak{J},\mathcal{K}),(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ be SSs over U. Then,*

(1) $(\mathfrak{J},\mathcal{K})\vee_\lambda[(\mathfrak{S},\mathcal{Z})\wedge(\mathfrak{Q},\mathcal{C})]\,\tilde{\subseteq}_L\,[(\mathfrak{J},\mathcal{K})\vee_\lambda(\mathfrak{S},\mathcal{Z})]\vee[(\mathfrak{J},\mathcal{K})\vee_\lambda(\mathfrak{Q},\mathcal{C})]$

(2) $(\mathfrak{J},\mathcal{K})\vee_\lambda[(\mathfrak{S},\mathcal{Z})\vee(\mathfrak{Q},\mathcal{C})]\,\tilde{\subseteq}_L\,[(\mathfrak{J},\mathcal{K})\vee_\lambda(\mathfrak{S},\mathcal{Z})]\wedge\big[(\mathfrak{J},\mathcal{K})\vee_{\lambda_*}(\mathfrak{Q},\mathcal{C})\big]$

Proof 26. (1) Let $(\mathfrak{S},\mathcal{Z})\wedge(\mathfrak{Q},\mathcal{C})=(\mathfrak{E},\mathcal{Z}\times\mathcal{C})$, where for all $(z,c)\in\mathcal{Z}\times\mathcal{C}$, $\mathfrak{E}(z,c)=\mathfrak{S}(z)\cap\mathfrak{Q}(c)$. Let $(\mathfrak{J},\mathcal{K})\vee_\lambda(\mathfrak{E},\mathcal{Z}\times\mathcal{C})=(\mathfrak{R},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C}))$, where for all $\big(\mathscr{k},(z,c)\big)\in\mathcal{K}\times(\mathcal{Z}\times\mathcal{C})$,

$$\mathfrak{R}\big(\mathscr{k},(z,c)\big)=\mathfrak{J}(\mathscr{k})\cup[\mathfrak{S}(z)\cap\mathfrak{Q}(c)]'=\mathfrak{J}(\mathscr{k})\cup\big[\mathfrak{S}'^{(z)}\cup\mathfrak{Q}'^{(c)}\big].$$

Assume that $(\mathfrak{J},\mathcal{K})\vee_\lambda(\mathfrak{S},\mathcal{Z})=(\mathcal{H},\mathcal{K}\times\mathcal{Z})$ and $(\mathfrak{J},\mathcal{K})\vee_\lambda(\mathfrak{Q},\mathcal{C})=(\mathcal{M},\mathcal{K}\times\mathcal{C})$, where for all $(\mathscr{k},z)\in\mathcal{K}\times\mathcal{Z}$, $\mathcal{H}(\mathscr{k},z)=\mathfrak{J}(\mathscr{k})\cup\mathfrak{S}'(z)$ and for all $(\mathscr{k},c)\in\mathcal{K}\times\mathcal{C}$, $\mathcal{M}(\mathscr{k},c)=\mathfrak{J}(\mathscr{k})\cup\mathfrak{Q}'(c)$. Let $(\mathcal{H},\mathcal{K}\times\mathcal{Z})\vee(\mathcal{M},\mathcal{K}\times\mathcal{C})=\big(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C})\big)$, where for all $\big((\mathscr{k},z),(\mathscr{k},c)\big)\in(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C})$,

$$\beta\big((\mathscr{k},z),(\mathscr{k},c)\big)=[\mathfrak{J}(\mathscr{k})\cup\mathfrak{S}'(z)]\cup[\mathfrak{J}(\mathscr{k})\cup\mathfrak{Q}'(c)].$$

Here, for all $\big(\mathscr{k},(z,c)\big)\in\mathcal{K}\times(\mathcal{Z}\times\mathcal{C})$, there exists $((\mathscr{k},z),(\mathscr{k},c))\in(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C})$ such that $\mathfrak{R}\big(\mathscr{k},(z,c)\big)=\mathfrak{J}(\mathscr{k})\cup[\mathfrak{S}'(z)\cup\mathfrak{Q}'(c)]=[\mathfrak{J}(\mathscr{k})\cup\mathfrak{S}'(z)]\cup[\mathfrak{J}(\mathscr{k})\cup\mathfrak{Q}'(c)]=\beta\big((\mathscr{k},z),(\mathscr{k},c)\big)$. This completes the proof. □

It is obvious that the L-subset in Proposition 23. can not be L-equality with the following example:

Example 2. *Let $E=\{\ell_1,\ell_2,\ell_3,\ell_4\}$ be the PS, $\mathcal{K}=\{\ell_2,\ell_3\}$, $\mathcal{Z}=\{\ell_1\}$, and $\mathcal{C}=\{\ell_4\}$, be the subsets of E, $U=\{\rlap{/}f_1,\rlap{/}f_2,\rlap{/}f_3,\rlap{/}f_4,\rlap{/}f_5\}$ be the universal set, $(\mathfrak{J},\mathcal{K})$, $(\mathfrak{S},\mathcal{Z})$ and $(\mathfrak{Q},\mathcal{C})$ ve SSs over U such that $(\mathfrak{J},\mathcal{K})=\{(\ell_2,\{\rlap{/}f_3,\rlap{/}f_4\}),(\ell_3,\{\rlap{/}f_2,\rlap{/}f_3\})\}$, $(\mathfrak{S},\mathcal{Z})=\{(\ell_1,U)\}$ and $(\mathfrak{Q},\mathcal{C})=\{(\ell_4,\{\rlap{/}f_2,\rlap{/}f_3,\rlap{/}f_4\})\}$. We show that*

$$(\mathfrak{J},\mathcal{K})\vee_\lambda[(\mathfrak{S},\mathcal{Z})\wedge(\mathfrak{Q},\mathcal{C})]\neq_L[(\mathfrak{J},\mathcal{K})\vee_\lambda(\mathfrak{S},\mathcal{Z})]\vee[(\mathfrak{J},\mathcal{K})\vee_\lambda(\mathfrak{Q},\mathcal{C})].$$

Let $(\mathfrak{S},\mathcal{Z})\wedge(\mathfrak{Q},\mathcal{C})=(\mathfrak{E},\mathcal{Z}\times\mathcal{C})$, where

$$(\mathfrak{S},\mathcal{Z})\wedge(\mathfrak{Q},\mathcal{C})=(\mathfrak{E},\mathcal{Z}\times\mathcal{C})=\big\{\big((\ell_1,\ell_4),\{\rlap{/}f_2,\rlap{/}f_3,\rlap{/}f_4\}\big)\big\}.$$

Assume that $(\mathfrak{J},\mathcal{K})\vee_\lambda(\mathfrak{E},\mathcal{Z}\times\mathcal{C})=\big(\mathfrak{M},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C})\big)$, where

$$\big(\mathfrak{M},\mathcal{K}\times(\mathcal{Z}\times\mathcal{C})\big)=\Big(\Big(\big(\ell_2,(\ell_1,\ell_4)\big),\{\rlap{/}f_1,\rlap{/}f_3,\rlap{/}f_4,\rlap{/}f_5\}\Big),\Big(\big(\ell_3,(\ell_1,\ell_4)\big),\{\rlap{/}f_1,\rlap{/}f_2,\rlap{/}f_3,\rlap{/}f_5\}\Big)\Big).$$

Let $(\mathfrak{J},\mathcal{K})\vee_\lambda(\mathfrak{S},\mathcal{Z})=(\mathfrak{E},\mathcal{K}\times\mathcal{Z})$, where

$$(\mathfrak{E},\mathcal{K}\times\mathcal{Z})=\big\{\big((\ell_2,\ell_1),\{\rlap{/}f_3,\rlap{/}f_4\}\big),\big((\ell_3,\ell_1),\{\rlap{/}f_2,\rlap{/}f_3\}\big)\big\}.$$

Suppose that $(\mathfrak{J},\mathcal{K})\vee_\lambda(\mathfrak{Q},\mathcal{C})=(\mathfrak{R},\mathcal{K}\times\mathcal{C})$, where

$$(\mathfrak{R},\mathcal{K}\times\mathcal{C})=\big[\big((\ell_2,\ell_4),\{\rlap{/}f_1,\rlap{/}f_3,\rlap{/}f_4,\rlap{/}f_5\}\big),\big((\ell_3,\ell_4),\{\rlap{/}f_1,\rlap{/}f_2,\rlap{/}f_3,\rlap{/}f_5\}\big)\big].$$

Let $(\mathfrak{E},\mathcal{K}\times\mathcal{Z})\vee(\mathfrak{R},\mathcal{K}\times\mathcal{C})=\big(\beta,(\mathcal{K}\times\mathcal{Z})\times(\mathcal{K}\times\mathcal{C})\big)$. Then,

$$\big(\beta, (\mathcal{K} \times \mathcal{Z}) \times (\mathcal{K} \times \mathcal{C})\big) = \left\{ \begin{matrix} \Big(\big((\ell_2, \ell_1), (\ell_2, \ell_4)\big), \{\not{f}_1, \not{f}_3, \not{f}_4, \not{f}_5\}\Big), \Big(\big((\ell_2, \ell_1), (\ell_3, \ell_4)\big), \{\not{f}_1, \not{f}_2, \not{f}_3, \not{f}_4, \not{f}_5\}\Big), \\ \Big(\big((\ell_3, \ell_1), (\ell_2, \ell_4)\big), \{\not{f}_1, \not{f}_2, \not{f}_3, \not{f}_4, \not{f}_5\}\Big), \Big(\big((\ell_3, \ell_1), (\ell_3, \ell_4)\big), \{\not{f}_1, \not{f}_2, \not{f}_3, \not{f}_5\}\Big) \end{matrix} \right\}.$$

Thus, $\beta\big((\ell_2, \ell_1), (\ell_3, \ell_4)\big) \neq \mathfrak{M}\big(\ell_2, (\ell_1, \ell_4)\big)$, $\beta\big((\ell_2, \ell_1), (\ell_3, \ell_4)\big) \neq \mathfrak{M}\big(\ell_3, (\ell_1, \ell_4)\big)$, $\beta\big((\ell_3, \ell_1), (\ell_2, \ell_4)\big) \neq \mathfrak{M}\big(\ell_2, (\ell_1, \ell_4)\big)$, $\beta\big((\ell_3, \ell_1), (\ell_2, \ell_4)\big) \neq \mathfrak{M}\big(\ell_3, (\ell_1, \ell_4)\big)$, implying that $\big(\beta, (\mathcal{K} \times \mathcal{Z}) \times (\mathcal{K} \times \mathcal{C})\big) \tilde{\nsubseteq}_{\mathrm{L}} \big(\mathfrak{M}, \mathcal{K} \times (\mathcal{Z} \times \mathcal{C})\big)$. Hence, $\big(\beta, (\mathcal{K} \times \mathcal{Z}) \times (\mathcal{K} \times \mathcal{C})\big) \neq_{\mathrm{L}} \big(\mathfrak{M}, \mathcal{K} \times (\mathcal{Z} \times \mathcal{C})\big)$.

5. *Int-uni* decision-making method applied to soft lambda-product

The *int-uni* decision-making approach is applied in this section by applying the *int-uni* operator and *int-uni* decision function developed by Çağman and Enginoğlu [11] to the soft lambda-product.

Throughout this section, all the soft lambda-products (V_λ) of the SSs over U are assumed to be contained in the set $\mathrm{V}_\lambda(\mathrm{U})$, and the approximation function of the soft lambda-product of $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$, that is $(\mathfrak{J}, \mathcal{K})\mathrm{V}_\lambda(\mathfrak{S}, \mathcal{Z})$

$$\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{S}_{\mathcal{Z}}: \mathcal{K}\mathrm{x}\mathcal{Z} \to \mathrm{P(U)}.$$

where $(\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{S}_{\mathcal{Z}})(\mathscr{k}, z) = \mathfrak{J}(\mathscr{k}) \cup \mathfrak{S}'(z)$ for all $(\mathscr{k}, z) \in \mathcal{K}\mathrm{x}\mathcal{Z}$.

Definition 16. *Let $(\mathfrak{J}, \mathcal{K})$ and $(\mathfrak{S}, \mathcal{Z})$ be SS over U. Then, int-uni operators for soft lambda-product, denoted by $int_x uni_y$ and $int_y uni_x$ are defined respectively as*

$$int_x uni_y: \mathrm{V}_\lambda \to \mathrm{P(U)},\ int_x uni_y\,(\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{S}_{\mathcal{Z}}) = \cap_{\mathscr{k}\in\mathcal{K}}(\cup_{z\in \mathcal{Z}}(\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{S}_{\mathcal{Z}})(\mathscr{k}, z))),$$

$$int_y uni_x: \mathrm{V}_\lambda \to \mathrm{P(U)},\ int_y uni_x\,(\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{S}_{\mathcal{Z}}) = \cap_{z\in \mathcal{Z}}(\cup_{\mathscr{k}\in\mathcal{K}}(\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{S}_{\mathcal{Z}})(\mathscr{k}, z))).$$

Definition 17. *[11] Let $(\mathfrak{J}, \mathcal{K})V_\lambda(\mathfrak{S}, \mathcal{Z}) \in V_\lambda(U)$. Then, int-uni decision function for soft lambda-product, denoted by int-uni are defined by*

$$int\text{-}uni: \mathrm{V}_\lambda \to \mathrm{P(U)},\ int\text{-}uni\,(\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{S}_{\mathcal{Z}}) = int_x uni_y\,(\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{S}_{\mathcal{Z}}) \cup int_y uni_x(\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{S}_{\mathcal{Z}}).$$

The values *int-uni*$(\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{S}_{\mathcal{Z}})$ is a subset of U called *int-uni* decision set of $\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{S}_{\mathcal{Z}}$.

The *int-uni* decision-making approach may be used in the following ways to choose the best set of alternatives while staying focused on the current issue given a set of parameters and options:

Step 1: From the parameter collection, choose feasible subsets.

Step 2: Create the SSs for every parameter sets.

Step 3: Determine the SSs' soft lambda-product.

Step 4: Create the result of *int-uni* decision set.

This method demonstrates the value of SS theory in handling decision-making scenarios by enabling its application to the *int-uni* decision-making dilemma, particularly in the setting of soft lambda-product.

Example 3. *A private teaching institution has announced a recruitment process to form a young and dynamic team of teachers. Initially, applications are reviewed to ensure they meet the required qualifications for the position, with ineligible applications being disqualified. Due to the large number of candidates remaining after this preliminary elimination, the institution has decided to adopt a two-stage* *evaluation process.*

In this process:

1) Stage 1: Mrs. Nazlan, a representative from the Human Resources department, will eliminate candidates based on their interview performance and exam results.
2) Stage 2: The candidates who pass the first stage will undergo a comprehensive training program, and those who successfully complete it will qualify to join the institution's professional teaching team.

During the evaluation, Mrs. Nazlan will identify:

- Parameters she DOES wish to see in the candidates to be eliminated: Traits or deficiencies that make a candidate unsuitable.
- Parameters she absolutely DOES NOT want to see in candidates to be eliminated: Key characteristics that make a candidate viable for further consideration.

Mrs. Nazlan will use the int-uni decision-making method on soft lambda-product to guide her selection. Let the set of candidates whose applications have been validated for the teacher recruitment process be: $U = \{r_1, r_2, \ldots, r_{35}\}$. Let the set of parameters to be used for identifying the teachers to be eliminated be Let the set of parameters used to identify the teachers to be eliminated be represented as $\{a_1, a_2, \ldots, a_{10}\}$. Each parameter a_i, where $i \in \{1, 2, \ldots, 10\}$ corresponds to the following descriptions:

- a_1: "Intolerant and impatient"
- a_2: "Having sufficient expertise in the field and adequate general knowledge"
- a_3: "Having ineffective classroom management skills"
- a_4: "Strong communication skills"
- a_5: "Taking individual differences into account in education"
- a_6: "Having insufficient teaching skills."
- a_7: "Being open to innovations and developments, continuously renewing oneself"
- a_8: "Not encouraging and supportive"
- a_9: "Not being cheerful, humorous, or affectionate"
- a_{10}: "Having poor diction."

To address the teacher selection process, we can apply the soft lambda-product method in the following manner:

Step 1: Determining the Sets of Parameters

Mrs. Nazlan, the decision-maker, selects parameters from the existing set that define the characteristics of candidates to be eliminated.

- Parameters that are preferred in candidates to be eliminated:

These are undesirable traits or deficiencies that make a candidate unsuitable for selection, but appropriate for elimination.

- Parameters that must NOT be present in eliminated candidates:

These represent essential qualities or skills required in a teacher, and their absence would disqualify a candidate.

By organizing these parameters into two sets, the selection process ensures clarity and alignment with the decision-maker's priorities. The parameter sets are as follows:

- $\mathcal{K} = \{a_1, a_3, a_6, a_8\}$
- $\mathcal{Z} = \{a_2, a_4, a_5, a_7\}$,

respectively.

Step 2: Constructing the SSs by Using the PSs Determined in Step 1.

Using these parameter sets, the decision-maker constructs the SSs $(\mathfrak{J},\mathcal{K})$ and $(\mathfrak{J},\mathcal{Z})$, respectively

$$(\mathfrak{J},\mathcal{K})=\{(a_1,\{r_2,r_6,r_7,r_{11},r_{13},r_{17},r_{19},r_{27},r_{31},r_{33}\}),(a_3,\{r_6,r_{10},r_{11},r_{18},r_{19},r_{21},r_{22},r_{30},r_{32},r_{33}\}),$$

$$(a_6,\{r_1,r_3,r_6,r_{10},r_{13},r_{18},r_{19},r_{25},r_{28},r_{32}\}),\left(a_8,\{r_1,r_3,r_6,r_{10},r_{11},r_{13},r_{17},r_{22},r_{25},r_{27},r_{29},r_{33}\}\right\},$$

$$(\mathfrak{J},\mathcal{Z})=\{(a_2,\{r_9,r_{14},r_{17},r_{23},r_{25},r_{27}\}),(a_4,\{r_2,r_3,r_7,r_9,r_{11},r_{14},r_{17},r_{21},r_{23},r_{30},r_{31},r_{32}\}),$$

$$(a_5,\{r_1,r_3,r_{14},r_{19},r_{20},r_{23},r_{27},r_{32}\}),(a_7,\{r_7,r_{14},r_{19},r_{20},r_{23},r_{27},r_{29},r_{32},r_{35})\}\}.$$

The SS $(\mathfrak{J},\mathcal{K})$ represents a set of candidates to be eliminated due to undesirable parameters in $\mathcal{K}$, while $(\mathfrak{J},\mathcal{Z})$ represents a set of candidates that are closer to the ideal by possessing the highly desired parameters in $\mathcal{Z}$. The process of constructing these sets involves assigning weights to the parameters and evaluating their importance in the decision-making process, ensuring that the elimination process is both balanced and justifiable. It is important to note that Mrs. Nazlan's task is specifically to eliminate the candidates based on these criteria.

Step 3: Determine the V_λ-product of SSs:

$$\mathfrak{J}_{\mathcal{K}}\mathrm{V}_\lambda\mathfrak{J}_{\mathcal{Z}}=$$

$$\{\left((a_1,a_2),\left\{\begin{matrix}r_1,r_2,r_3,r_4,r_5,r_6,r_7,r_8,r_{10},r_{11},r_{12},r_{13},r_{15},r_{16},r_{17},r_{18},r_{19},r_{20},r_{21},r_{22},\\ r_{24},r_{26},r_{27},r_{28},r_{29},r_{30},r_{31},r_{32},r_{33},r_{34},r_{35}\end{matrix}\right\}\right),$$

$$\left((a_1,a_4),\left\{\begin{matrix}r_1,r_2,r_4,r_5,r_6,r_7,r_8,r_{10},r_{11},r_{12},r_{13},r_{15},r_{16},r_{17},r_{18},r_{19},r_{20},r_{22},\\ r_{24},r_{25},r_{26},r_{27},r_{28},r_{29},r_{31},r_{33},r_{34},r_{35}\end{matrix}\right\}\right),$$

$$\left((a_1,a_5),\left\{\begin{matrix}r_2,r_4,r_5,r_6,r_7,r_8,r_9,r_{10},r_{11},r_{12},r_{13},r_{15},r_{16},r_{17},r_{18},r_{19},r_{21},r_{22},\\ r_{24},r_{25},r_{26},r_{27},r_{28},r_{29},r_{30},r_{31},r_{33},r_{34},r_{35}\end{matrix}\right\}\right),$$

$$\left((a_1,a_7),\left\{\begin{matrix}r_1,r_2,r_3,r_4,r_5,r_6,r_7,r_8,r_9,r_{10},r_{11},r_{12},r_{13},r_{15},r_{16},r_{17},r_{18},r_{19},r_{21},r_{22},r_{24},r_{25},\\ r_{26},r_{27},r_{28},r_{30},r_{31},r_{33},r_{34}\end{matrix}\right\}\right),$$

$$\left((a_3,a_2),\left\{\begin{matrix}r_1,r_2,r_3,r_4,r_5,r_6,r_7,r_8,r_{10},r_{11},r_{12},r_{13},r_{15},r_{16},r_{18},r_{19},r_{20},r_{21},r_{22},r_{24},\\ r_{26},r_{28},r_{29},r_{30},r_{31},r_{32},r_{33},r_{34},r_{35}\end{matrix}\right\}\right),$$

$$\left((a_3,a_4),\left\{\begin{matrix}r_1,r_4,r_5,r_6,r_8,r_{10},r_{11},r_{12},r_{13},r_{15},r_{16},r_{18},r_{19},r_{20},r_{21},r_{22},r_{24},r_{25},\\ r_{26},r_{27},r_{28},r_{29},r_{30},r_{32},r_{33},r_{34},r_{35}\end{matrix}\right\}\right),$$

$$\left((a_3,a_5),\left\{\begin{matrix}r_2,r_4,r_5,r_6,r_7,r_8,r_9,r_{10},r_{11},r_{12},r_{13},r_{15},r_{16},r_{17},r_{18},r_{19},r_{21}r_{22},r_{24},r_{25},\\ r_{26},r_{28},r_{29},r_{30},r_{31},r_{32},r_{33},r_{34},r_{35}\end{matrix}\right\}\right),$$

$$\left((a_3,a_7),\left\{\begin{matrix}r_1,r_2,r_3,r_4,r_5,r_6,r_8,r_9,r_{10},r_{11},r_{12},r_{13},r_{15},r_{16},r_{17},r_{18},r_{19},r_{21},r_{22},r_{24},r_{25},\\ r_{26},r_{28},r_{30},r_{31},r_{32},r_{33},r_{34}\end{matrix}\right\}\right),$$

$$\left((a_6,a_2),\left\{\begin{matrix}r_1,r_2,r_3,r_4,r_5,r_6,r_7,r_8,r_{10},r_{11},r_{12},r_{13},r_{15},r_{16},r_{18},r_{19},r_{20},r_{21},r_{22},r_{24},r_{25},\\ r_{26},r_{28},r_{29},r_{30},r_{31},r_{32},r_{33},r_{34},r_{35}\end{matrix}\right\}\right),$$

$$\left((a_6,a_4),\left\{\begin{matrix}r_1,r_3,r_4,r_5,r_6,r_8,r_{10},r_{12},r_{13},r_{15},r_{16},r_{18},r_{19},r_{20},r_{22},r_{24},r_{25},\\ r_{26},r_{27},r_{28},r_{29},r_{32},r_{33},r_{34},r_{35}\end{matrix}\right\}\right),$$

$$\left((a_6, a_5), \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}\right),$$

$$\left((a_6, a_7), \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{28}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34} \end{matrix}\right\}\right),$$

$$\left((a_8, a_2), \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}\right),$$

$$\left((a_8, a_4), \left\{\begin{matrix} r_1, r_3, r_4, r_5, r_6, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}\right),$$

$$\left((a_8, a_5), \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}\right),$$

$$\left((a_8, a_7), \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{33}, r_{34} \end{matrix}\right\}\right)\}.$$

Step 4: Determine the set of *int-uni*($\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z}$):

$$int_{k} - uni_{z}(\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z}) = \cap_{k \in \mathcal{K}} \left(\cup_{z \in Z}\left((\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(k, z)\right)\right)$$

We first determine $\cup_{z \in Z} \left((\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(k, z)\right)$:

$$(\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_1, a_2) \cup (\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_1, a_4) \cup (\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_1, a_5) \cup (\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_1, a_7)$$

$$= \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, \\ r_{24}, r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{22}, \\ r_{24}, r_{25}, r_{26}, r_{27}, r_{28}, r_{29}, r_{31}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, \\ r_{24}, r_{25}, r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{30}, r_{31}, r_{33}, r_{34} \end{matrix}\right\}$$

$$= \begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}.$$

$$(\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_3, a_2) \cup (\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_3, a_4) \cup (\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_3, a_5) \cup (\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_3, a_7)$$

$$= \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, \\ r_{26}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} r_1, r_4, r_5, r_6, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21} r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{28}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}.$$

$$(\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_6, a_2) \cup (\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_6, a_4) \cup (\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_6, a_5) \cup (\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{Z})(a_6, a_7)$$

$$= \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_8, \mathscr{r}_{10}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_9, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_8, \mathscr{r}_9, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{28}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_9, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}.$$

$$(\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}})(a_8, a_2) \cup \mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}})(a_8, a_4) \cup (\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}})(a_8, a_5) \cup (\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}})(a_8, a_7)$$

$$= \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_8, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_9, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_8, \mathscr{r}_9, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{33}, \mathscr{r}_{34} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_9, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}.$$

Thus,

$$(int_{\mathscr{k}} - uni_{z})(\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}}) = \cap_{\mathscr{k} \in \mathcal{K}} \left(\cup_{z \in \mathcal{Z}} \big(\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}})(\mathscr{k}, z) \big) \right) =$$

$$\left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_9, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}$$

$$\cap \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_9, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}$$

$$\cap \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_9, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}$$

$$\cap \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_9, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_9, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \mathscr{r}_{25}, \\ \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}.$$

is obtained.

$$(int_{z} - uni_{\mathscr{k}})(\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}}) = \cap_{z \in \mathcal{Z}} \left(\cup_{\mathscr{k} \in \mathcal{K}} \big((\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}})(\mathscr{k}, z) \big) \right).$$

We first determine $\cup_{\mathscr{k} \in \mathcal{K}} \big((\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}})(\mathscr{k}, z) \big)$:

$$(\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}})(a_1, a_2) \cup (\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}})(a_3, a_2) \cup (\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}})(a_6, a_2) \cup (\mathfrak{I}_{\mathcal{K}} \mathrm{V}_{\lambda} \mathfrak{I}_{\mathcal{Z}})(a_8, a_2)$$

$$= \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{17}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{21}, \mathscr{r}_{22}, \\ \mathscr{r}_{24}, \mathscr{r}_{26}, \mathscr{r}_{27}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} \mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, \mathscr{r}_4, \mathscr{r}_5, \mathscr{r}_6, \mathscr{r}_7, \mathscr{r}_8, \mathscr{r}_{10}, \mathscr{r}_{11}, \mathscr{r}_{12}, \mathscr{r}_{13}, \mathscr{r}_{15}, \mathscr{r}_{16}, \mathscr{r}_{18}, \mathscr{r}_{19}, \mathscr{r}_{20}, \mathscr{r}_{21}, \mathscr{r}_{22}, \mathscr{r}_{24}, \\ \mathscr{r}_{26}, \mathscr{r}_{28}, \mathscr{r}_{29}, \mathscr{r}_{30}, \mathscr{r}_{31}, \mathscr{r}_{32}, \mathscr{r}_{33}, \mathscr{r}_{34}, \mathscr{r}_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}.$$

$$(\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_1, a_4) \cup (\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_3, a_4) \cup (\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_6, a_4) \cup (\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_8, a_4)$$

$$= \left\{\begin{matrix} r_1, r_2, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{22}, \\ r_{24}, r_{25}, r_{26}, r_{27}, r_{28}, r_{29}, r_{31}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} r_1, r_4, r_5, r_6, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_1, r_3, r_4, r_5, r_6, r_8, r_{10}, r_{12}, r_{13}, r_{15}, r_{16}, r_{18}, r_{19}, r_{20}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_1, r_3, r_4, r_5, r_6, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}.$$

$$(\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_1, a_5) \cup (\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_3, a_5) \cup (\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_6, a_5) \cup (\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_8, a_5)$$

$$= \left\{\begin{matrix} r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, \\ r_{24}, r_{25}, r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} r_2, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21} r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}.$$

$$(\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_1, a_7) \cup (\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_3, a_7) \cup (\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_6, a_7) \cup (\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(a_8, a_7)$$

$$= \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{30}, r_{31}, r_{33}, r_{34} \end{matrix}\right\}.$$

$$\cup \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{28}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{28}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34} \end{matrix}\right\}$$

$$\cup \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{33}, r_{34} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34} \end{matrix}\right\}.$$

is obtained. Therefore,

$$(int_z - uni_k)(\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}}) = \cap_{z \in \mathcal{Z}} \left(\cup_{k \in \mathcal{K}} \left((\mathfrak{J}_{\mathcal{K}} \vee_{\lambda} \mathfrak{J}_{\mathcal{Z}})(k, z) \right) \right) =$$

$$\left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cap \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cap \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cap \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}.$$

Thus,

$$int\text{-}uni(\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{\mathcal{Z}}) = [int_{\mathscr{k}} - uni_{\mathscr{z}}(\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{\mathcal{Z}})] \cup [int_{\mathscr{z}} - uni_{\mathscr{k}}(\mathfrak{I}_{\mathcal{K}} \vee_{\lambda} \mathfrak{I}_{\mathcal{Z}})]$$

$$= \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$\cup = \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{15}, r_{16}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{24}, r_{25}, \\ r_{26}, r_{27}, r_{28}, r_{29}, r_{30}, r_{31}, r_{32}, r_{33}, r_{34}, r_{35} \end{matrix}\right\}.$$

Therefore, in the teacher recruitment process at the private teaching institution, out of the 35 candidates whose applications were accepted, 33 were eliminated in the first stage. The remaining candidates, $\{r_{14}, r_{23}\}$ were enrolled in a comprehensive training program and subsequently earned the right to join the institution's professional teacher team.

6. Conclusion

The "soft lambda-product," a novel kind of soft product derived from Molodtsov's soft set theory, was presented in this paper. We analyzed its algebraic features in detail and gave an example with respect to several kinds of soft subsets and equalities, such as M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality. Additionally, we looked at the soft lambda-product's distributional rules over several soft set operations. To choose the best components from the various possibilities, we finally applied the soft decision-making strategy, which streamlines the process by doing away with the necessity for rough or fuzzy soft sets. An example shows how well it works in a variety of disciplines. Numerous applications, including innovative soft set-based cryptography algorithms and fresh approaches to decision-making, are made possible by this work. Future studies might suggest more soft product operations and investigate basic characteristics associated with different soft equal relations in order to theoretically and practically enhance the soft set literature.

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