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Article

Parametric rough bi-level multi-objective fractional programming problems

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Abstract: The parametric rough bi-level multi-objective fractional programming problem (PRBL-MOFPP) is investigated in this article. In the right-hand aspect of the rough set of constraints, the suggested PRBL-MOFPP has a scalar parameter. The PRBL-MOFPP is converted into two issues congruent to the upper and lower approximation models (UAM and LAM) in the first phase. Both UAM and LAM are solved using the fuzzy goal programming (FGP) technique. The parametric UAM and LAM were formulated in the second phase, and the Lagrangian function for UAM and LAM was derived. Furthermore, both models were subjected to Karush-Kuhn-Tucker (KKT) optimality conditions. Finally, the surely and possibly stable set of the first kind (SSFK) are studied. An algorithm for determining the SSFK for PRBL-MOFPP, as well as numerical examples, are exhibited.

Keywords: bi-level optimization; fractional programming; parametric uncertainty; rough set; FGP approach

1. Introduction

Parametric programming is an effective tool in mathematical programming. The fundamentals of parametric convex programming have been introduced in [1,2]. Osman et al. in [3] presented a parametric linear goal programming (GP) issue.

The researchers are faced with a significant challenge by ambiguity and handling insufficient data. Numerous mathematical theories [4–9] are introduced to solve such issues. Rough set theory (RST) is a powerful modeling technique for such uncertain circumstances [7,10–13]. Pawlak's RST is the most recent theory for the collaborative handling of ambiguity and uncertainty [11,12]. It provides a strong theoretical foundation for talking about the knowledge that can categorize objects. In a crisp and ordinary set, an object is precisely determined based on all available information, whereas in RST, it is only roughly determined. A pair of precise notions known as the lower and upper approximations of the vague concept are used in RST to replace any vague concepts. A lower approximation for a hazy idea X includes all objects that unquestionably belong to it, whereas an upper approximation includes all objects that might conceivably belong to it [12,14,15].

The bi-level programming problem (BLPP) is used naturally in many critical resource plannings, management problems, and policymaking areas. BLPP is a powerful tool for dealing with hierarchical decision-making challenges that are widespread in government initiatives, financial systems, logistics networks, agriculture, vehicle routing issues, etc. [6,16–23,24]. These issues are structured as multi-objective programming challenges. Numerous approaches have been put out to address issues like those in [24–28].

The optimization of one or more ratios of functions subject to a set of constraints was covered in the field of fractional programming (FP). In recent decades, FP evolved as one of the planning tools. It has been used in a variety of sectors, including engineering, business, banking, and economics [29–32]. The basic methods and applications of the bi-level multi-objective fractional programming problems (BL-MOFP) have been discussed in the literature [17,29,33–35]. Recently, numerous studies have been done on BL-MOFPs under uncertainty [29,36–39]. Ranarahu et al. suggested a strategy for dealing with fuzzy probabilistic multiple objective BLPP [40]. Baky et al. created the FGP algorithm to solve fuzzy BL-MOPP problems [20]. Ren devised a way for dealing with the totally fuzzy BL-MOPP using interval programming concepts [41]. The parametric fuzzy form of the multi-level optimization problem (ML-OP) was demonstrated by Osman et al. [33]. Arora et al. released an interactive FGP strategy for BL-PP [24]. ML-OP was studied by Chen and Chen [21]. Fuzzy integer BL-MOPP was demonstrated by Youness et al. [32]. Saad et al. investigated the rough interval three-level quadratic programming topic [42]. Elsisy and Elsayed offered three scenarios for dealing with fuzzy rough BL-MOFP problems [8]. Gumus and Floudas developed an innovative approach for solving the nonlinear BL-MOPP to global optimality [43]. El Sayed et al. investigated an M-TOPSIS methodology to resolve unidentified fractional ML-OP [25]. El Sayed and Farahat presented research on the Achievement Stability Set for Parametric FGP issues [36]. Osman et al. investigated the uncertainty for nonlinear ML-OP [33,44]. Hamzehee et al. [15] and Jivping Xu [45] presented a linear programming issue where part or all the choice parameters are rough intervals.

Motivation and contribution

Mathematical programming has a crucial principle that applies to parametric analysis. The goal of the parametric analysis is to determine how the behavior of the efficient set or the optimal value differs as data is modified. As a result, it frequently serves as a talking point when addressing ambiguity [1,2]. The SSFK are the quantities and relationships between different parameters that have the same solutions [1,2,46]. The earlier research that was given away in this era concentrated on stabilizing the solution. In addition, the roughness associated with the collection of constraints known as the rough environment is the type of uncertainty that receives crucial attention in realistic optimization problems. As the feasible area is rough, Farahat and El Sayed presented the parametric rough FGP issue [47,48-50]. Osman et al. demonstrated a parametric fuzzy BL-MOFP with a goal parameter and fuzziness in constraints [3]. None of the previous studies proposed the BL-MOFP issue with a parametric rough environment. Moreover, the SSFK for that model has not been introduced before.

In this study, the SSFK of the PRBL-MOFPP is considered. The proposed PRBL-MOFPP has a scalar parameter in the right-hand side of the rough set of constraints. In the first phase, the PRBL-MOFPP is transformed into two issues corresponding to the UAM and LAM. Pal's method is applied to linearize the membership goals for the UAM and the LAM. Then the FGP approach is utilized to find a compromise solution for both UAM and LAM. In the second phase, the parametric UAM and LAM were formulated, and then the Lagrangian function for UAM and LAM was obtained. Moreover, the KKT optimality conditions are applied to both models. At the end we define the surely and possibly SSFK for the PRBL-MOFPP. A procedure for obtaining the progressing SSFK for PRBL-MOFPP, as well as an illustrative numerical example, are exhibited.

The remainder of this article is structured as follows: Sect. 2 presents problem formulation and methodology for the PRBL-MOFPPs. In Sect. 3, the surely and possibly SSFK for PRBL-MOFPP was introduced. Sect. 4 incorporates an algorithm for investigating the SSFK for PRBL-MOFPP. A numerical example was given in Sect. 5. A few findings are included at the end.

2. Problem formulation and methodology

The PRBL-MOFPP can be formulated as [5,17,37]:

[1st Level]

$$\underbrace{max}_{x_1}\ F_1(x) = \frac{c_{1q}^T x + \alpha_{1q}}{d_{1q}^T x + \beta_{1q}}, \qquad q = 1,2,\dots,Q_1 \tag{1}$$

where x_2 solves

[2nd Level]

$$\underbrace{max}_{x_2}\ F_2(x) = \frac{c_{2q}^T x + \alpha_{2q}}{d_{2q}^T x + \beta_{2q}}, \qquad q = 1,2,\dots,Q_2 \tag{2}$$

subject to

$$x \in S \tag{3}$$

where

$$S_*(\gamma) \subseteq \mathrm{S} \subseteq S^*(\delta) \tag{4}$$

$$S_*(\gamma) = \left\{ x \in R^n; \sum_{j=1}^{n} a_{rj} x_j \le \gamma_r, \qquad r = 1,2\,, \qquad x \ge 0 \right\} \tag{5}$$

$$S^*(\delta) = \left\{ x \in R^n; \sum_{j=1}^{n} A_{rj} x_j \le \delta_r, \qquad r = 1,2, \qquad x \ge 0 \right\} \tag{6}$$

Also, $F_1 = (f_{11}, f_{12}, \dots, f_{1q_1})$, $F_2 = (f_{21}, f_{22}, \dots, f_{2q_2})$, are the objective functions of the first-level decision maker (FLDM) and second-level decision maker (SLDM), respectively. Notice that, $c_{11}, c_{12}, \dots, c_{1q_1}$; $c_{21}, c_{22}, \dots, c_{2q_2}$; $d_{11}, d_{12}, \dots, d_{1q_1}$; $d_{21}, d_{22}, \dots, d_{2q_2}$ are n-vectors, and $\alpha_{1r}, \alpha_{2r}, \beta_{1r}, \beta_{2r}$; are constants. The vector of decision variables $x = (x_1, x_2) \in R^n$ is partitioned between the two planners $x_1 \in R^{n_1}$; $x_2 \in R^{n_2}, n = n_1 + n_2$. Also, γ_r and δ_r are single scalar parameters represent the right-hand sides of $S_*(\gamma)$ the LAM and $S^*(\delta)$, the UAM of S [7,47].

To obtain the SSFK for problems (1)–(6), the SSFK for the following two parametric models, UAM and LAM, will be investigated. Thus, the UAM follows as [7,8,47]:

[1st Level]

$$\underbrace{max}_{x_1} \; F_1(x) = \frac{c_{1q}^T x + \alpha_{1q}}{d_{1q}^T x + \beta_{1q}}, \qquad q = 1,2,\dots,Q_1 \tag{7}$$

where x_2 solves

[2nd Level]

$$\underbrace{max}_{x_2} \; F_2(x) = \frac{c_{2q}^T x + \alpha_{2q}}{d_{2q}^T x + \beta_{2q}}, \qquad q = 1,2,\dots,Q_2 \tag{8}$$

subject to

$$S^*(\delta) = \left\{ x \in R^n; \sum_{j=1}^{n} A_{rj} x_j \le \delta_r, \qquad i = 1,2, \qquad x \ge 0 \right\} \tag{9}$$

The LAM of the PRBL-MOFPP follows as [7,47,48].

[1st Level]

$$\underbrace{max}_{x_1} \; F_1(x) = \frac{c_{1q}^T x + \alpha_{1q}}{d_{1q}^T x + \beta_{1q}}, \qquad q = 1,2,\dots,Q_1 \tag{10}$$

where x_2 solves

[2nd Level]

$$\underbrace{max}_{x_2} \; F_2(x) = \frac{c_{2q}^T x + \alpha_{2q}}{d_{2q}^T x + \beta_{2q}}, \qquad q = 1,2,\dots,Q_2 \tag{11}$$

subject to

$$S_*(\gamma) = \left\{ x \in R^n; \sum_{j=1}^{n} a_{rj} x_j \le \gamma_r, \qquad r = 1,2\,, \qquad x \ge 0 \right\} \tag{12}$$

2.1. FGP Approach for PRBL-MOFPP

To find a compromise solution, the FGP technique is used at $\delta = \delta^0, \gamma = \gamma^0$. Firstly, a fuzzy goal for each objective function is formulated by the membership function $\mu_{rq}\left(f_{rq}(x)\right), (r = 1,2), (q = 1,2,\dots,Q_r)$, at each level for UAM as [5,19,37,48,49,50]:

$$\mu_{rq}\left(f_{rq}(x)\right) = \begin{cases} 1, & if \quad f_{rq}(x) \ge u_{rq}^0, \\ \dfrac{f_{rq}(x) - g_{rq}^0}{u_{rq}^0 - g_{rq}^0}, & if \;\; g_{rq}^0 \le f_{rq}(x) \le u_{rq}^0, \;\; r = 1,2; \;\; q = 1,2,\dots,Q_r \\ 0, & if \quad f_{rq}(x) \le g_{rq}^0, \end{cases} \tag{13}$$

where the equivalent aspiration level is determined by taking each objective function's unique maximum, which is defined as:

$$u_{rq}^{0} = \underbrace{max}_{x \in S^{*}(\delta^{0})} \left(f_{rq}(x)\right), \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{14}$$

u_{rq}^{0} is the upper tolerance limit at $\delta = \delta^{0}$ for UAM. The unique minimum of each objective function is used to determine the related desire level [5,19]:

$$g_{rq}^{0} = \underbrace{min}_{x \in S^{*}(\delta^{0})} \left(f_{rq}(x)\right), \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{15}$$

g_{rq}^{0} is the lower tolerance limit at $\delta = \delta^{0}$ for UAM. Since one is the membership function with the highest degree. The membership goals for UAM at $\delta = \delta^{0}$ can be written as [5,24]:

$$\frac{f_{rq}(x) - g_{rq}^{0}}{u_{rq}^{0} - g_{rq}^{0}} + d_{rq}^{-} - d_{rq}^{+} = 1, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{16}$$

where $d_{rq}^{-},\ d_{rq}^{+} \geq 0$ with $d_{rq}^{-} \times d_{rq}^{+} = 0$, represent the under and over deviations respectively from the aspiration levels.

In the current FGP methodology, the sum of unwanted deviational variables is minimized to attain the target level. Thus, the FGP model of the UAM can be exhibited as [7]:

$$min\ Z_{rq}^{UAM} = \sum_{q=1}^{Q_1} w_{1q}^{-}\, d_{1q}^{-} + \sum_{q=1}^{Q_2} w_{2q}^{-} d_{2q}^{-} \tag{17}$$

subject to

$$\frac{f_{rq}^{UAM}(x) - g_{rq}^{0}}{u_{rq}^{0} - g_{rq}^{0}} + d_{rq}^{-} - d_{rq}^{+} = 1, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{18}$$

$$S^{*}(\delta^{0}) = \left\{ x \in R^{n}; \sum_{j=1}^{n} A_{rj} x_j \leq \delta_r^{0}, \qquad r = 1,2, \qquad x \geq 0 \right\} \tag{19}$$

$$x_{1v}^{UAM} = x_{1v}^{UAM^{*}} \qquad v = 1,2, \dots, n_1 \tag{20}$$

$$x, d_{rq}^{-}, d_{rq}^{+} \geq 0, \quad d_{rq}^{-} \times d_{rq}^{+} = 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{21}$$

where Z represents the achievement function and $w_{rq}^{-},\ r = 1,2;\ q = 1,2, \dots, Q_r$ indicate the relative significance of reaching the individual fuzzy goals' aspired levels; these values are obtained as [5,19]:

$$w_{rq}^{-} = \frac{1}{u_{rq}^{0} - g_{rq}^{0}}, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{22}$$

Similarly, for LAM at $\gamma = \gamma^{0}$, the FGP model of the LAM can be exhibited as:

$$min\ Z_{rq}^{LAM} = \sum_{q=1}^{Q_1} w_{1q}^{-}\, d_{1q}^{-} + \sum_{q=1}^{Q_2} w_{2q}^{-} d_{2q}^{-} \tag{23}$$

subject to

$$\frac{f_{rq}^{LAM}(x) - g_{rq}^{0}}{u_{rq}^{0} - g_{rq}^{0}} + d_{rq}^{-} - d_{rq}^{+} = 1, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{24}$$

$$S_*(\gamma^0) = \left\{ x \in R^n; \sum_{j=1}^{n} a_{rj} x_j \leq \gamma_r^0, \qquad r = 1,2, \qquad x \geq 0 \right\} \tag{25}$$

$$x_{1v}^{LAM} = x_{1v}^{LAM^*} \qquad v = 1,2, \dots, n_1 \tag{26}$$

$$x, d_{rq}^{-}, d_{rq}^{+} \geq 0, \quad d_{rq}^{-} \times d_{rq}^{+} = 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{27}$$

2.2. Linearization of membership goals

It is obvious that the membership goals in Equation (13) are intrinsically non-linear, which causes computing challenges in the solution phase. A linearization approach is used to prevent such issues. considered. Utilizing Pal et al. [32], the $\mu_{f_{rq}}$ modeled as:

$$\mu_{f_{rq}}\left(f_{rq}^{UAM}(x)\right) + d_{rq}^{-} - d_{rq}^{+} = 1, \qquad (r = 1,2), (q = 1, \dots, Q_r) \tag{28}$$

$$L_{rq}\left(f_{rq}^{UAM}(x)\right) - L_{rq} g_{rq}^{0} + d_{rq}^{-} - d_{rq}^{+} = 1, \qquad \text{where} \quad L_{rq} = \frac{1}{u_{rq}^{0} - g_{rq}^{0}} \tag{29}$$

$$f_{rq}^{UAM}(x) = \frac{c_{rq}^{T} x + \alpha_{rq}}{d_{rq}^{T} x + \beta_{rq}}, \qquad (r = 1,2), (q = 1, \dots, Q_r) \tag{30}$$

Considering the expression of $f_{ij}(x, \theta^0)$, the above goal in Equation (22) can be stated as:

$$L_{rq} \frac{c_{rq}^{T} x + \alpha_{rq}}{d_{rq}^{T} x + \beta_{rq}} - L_{rq} g_{rq}^{0} + d_{rq}^{-} - d_{rq}^{+} = 1 \tag{31}$$

$$L_{rq}\left[c_{rq}^{T} x + \alpha_{rq}\right] - L_{rq} g_{rq}^{0}\left[d_{rq}^{T} x + \beta_{rq}\right] + d_{rq}^{-}\left[d_{rq}^{T} x + \beta_{rq}\right] - d_{rq}^{+}\left[d_{rq}^{T} x + \beta_{rq}\right] = \left[d_{rq}^{T} x + \beta_{rq}\right]$$

$$L_{rq}\left[c_{rq}^{T} x + \alpha_{rq}\right] + d_{rq}^{-}\left[d_{rq}^{T} x + \beta_{rq}\right] - d_{rq}^{+}\left[d_{rq}^{T} x + \beta_{rq}\right] = \left(1 + L_{rq} g_{rq}^{0}\right)\left[d_{rq}^{T} x + \beta_{rq}\right]$$

$$L_{rq}\left[c_{rq}^{T} x + \alpha_{rq}\right] + d_{rq}^{-}\left[d_{rq}^{T} x + \beta_{rq}\right] - d_{rq}^{+}\left[d_{rq}^{T} x + \beta_{rq}\right] = L_{rq}^{0}\left[d_{rq}^{T} x + \beta_{rq}\right]$$

where $L_{rq}^{0} = \left(1 + L_{rq} g_{rq}^{0}\right)$,

$$\left[L_{rq} c_{rq}^{T} - L_{rq}^{0} d_{rq}^{T}\right] x + d_{rq}^{-}\left[d_{rq}^{T} x + \beta_{rq}\right] - d_{rq}^{+}\left[d_{rq}^{T} x + \beta_{rq}\right] = \left[L_{rq}^{0} \beta_{rq} - L_{rq} \alpha_{rq}\right]$$

$$C_{rq} x + d_{rq}^{-}\left[d_{rq}^{T} x + \beta_{rq}\right] - d_{rq}^{+}\left[d_{rq}^{T} x + \beta_{rq}\right] = G_{rq} \tag{32}$$

where

$$C_{rq} = \left[L_{rq} c_{rq}^{T} - L_{rq}^{0} d_{rq}^{T}\right] \text{ and } G_{rq} = \left[L_{rq}^{0} \beta_{rq} - L_{rq} \alpha_{rq}\right], \qquad r = 1,2; \quad q = 1, \dots, Q_r \tag{33}$$

Based on the variable change method by Pal et al. [32], the goal expression in Equation (35) can be linearized as follows. Letting $D_{rq}^{-} = d_{rq}^{-}\left[d_{rq}^{T} x + \beta_{rq}\right]$ and

$D_{rq}^{+} = d_{rq}^{+}\left[d_{rq}^{T}x + \beta_{rq}\right]$, subsequently, the linear model for articulation in Equation (25) is obtained as:

$$C_{rq}x + D_{rq}^{-} - D_{rq}^{+} = G_{rq} \tag{34}$$

with $D_{rq}^{-}, D_{rq}^{+} \geq 0$; and $D_{rq}^{-} \times D_{rq}^{+} = 0$, since $d_{rq}^{-}, d_{rq}^{+} \geq 0$, and $d_{rq}^{T}x + \beta_{rq} > 0$. Now, minimization of d_{rq}^{-} implies minimization of $D_{rq}^{-} = d_{rq}^{-}\left[d_{rq}^{T}x + \beta_{rq}\right]$, which is also non-linear. As a result of $d_{rq}^{-} \leq 1$, involvement in the solution, the following restriction is imposed in the problem:

$$\frac{D_{rq}^{-}}{\left[d_{rq}^{T}x + \beta_{rq}\right]} \leq 1 \tag{35}$$

The finalized FGP model for the UAM of the PRBL-MOFPP (7)–(9) becomes:

$$min\ \ Z_{rq}^{UAM} = \sum_{q=1}^{Q_1} w_{1q}^{-}d_{1q}^{-} + \sum_{q=1}^{Q_2} w_{2q}^{-}d_{2q}^{-} \tag{36}$$

subject to

$$\left[L_{rq}c_{rq}^{T} - L_{rq}^{0}d_{rq}^{T}\right]x + D_{rq}^{-} - D_{rq}^{+} = \left[L_{rq}^{0}\beta_{rq} - L_{rq}\alpha_{rq}\right] \quad r = 1,2;\ q = 1, \ldots, Q_r \tag{37}$$

$$\sum_{j=1}^{n} A_{rj}x_j \leq \delta_r^0, \qquad r = 1,2, \tag{38}$$

$$-d_{rq}^{T}x + D_{rq}^{-} \leq \beta_{rq}, \qquad (r = 1,2), (q = 1,2, \ldots, Q_r) \tag{39}$$

$$x_{1v}^{UAM} = x_{1v}^{UAM^*} \qquad v = 1,2, \ldots, n_1 \tag{40}$$

$$x_{rv}, D_{rq}^{-}, D_{rq}^{+} \geq 0, \quad D_{rq}^{-} \times D_{rq}^{+} = 0, \qquad (r = 1,2), (q = 1,2, \ldots, Q_r) \tag{41}$$

As a result, the above model provides a satisfactory solution $\boldsymbol{x}^*$ for the UAM of the PRBL-MOFPP.

Similarly, the final FGP model for the LAM of the PRBL-MOFPP (10)–(12) becomes.

$$min\ \ Z_{rq}^{LAM} = \sum_{q=1}^{Q_1} w_{1q}^{-}d_{1q}^{-} + \sum_{q=1}^{Q_2} w_{2q}^{-}d_{2q}^{-} \tag{42}$$

subject to

$$\left[L_{rq}c_{rq}^{T} - L_{rq}^{0}d_{rq}^{T}\right]x + D_{rq}^{-} - D_{rq}^{+} = \left[L_{rq}^{0}\beta_{rq} - L_{rq}\alpha_{rq}\right] \quad r = 1,2;\ \ q = 1, \ldots, Q_r \tag{43}$$

$$\sum_{j=1}^{n} a_{rj}x_j \leq \gamma_r^0, \qquad r = 1,2 \tag{44}$$

$$-d_{rq}^{T}x + D_{rq}^{-} \leq \beta_{rq}, \qquad (r = 1,2), (q = 1,2, \ldots, Q_r) \tag{45}$$

$$x_{1v}^{LAM} = x_{1v}^{LAM^*} \qquad v = 1,2, \dots, n_1 \tag{46}$$

$$\boldsymbol{x}_{rv}, D_{rq}^-, D_{rq}^+ \geq 0, \quad D_{rq}^- \times D_{rq}^+ = 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{47}$$

As a result, the above model provides a satisfactory solution $\boldsymbol{x}^*$ for the LAM of the PRBL-MOFPP.

Definition 1. *For any feasible* $x_1(x_1 \in S^*(\delta^0))$ *given by the FLDM if* $x_2(x_2 \in S^*(\delta^0))$ *is the Pareto optimal solution of the PRBL-MOFPP, then* (x_1, x_2) *is a feasible solution of the PRBL-MOFPP for the UAM.*

Definition 2. x^* *is a Pareto optimal solution of the PRBL-MOFPP for the UAM if there exist no other feasible value* $x \in S^*(\delta^0)$ *exist, such that* $f_{1q}^{UAM}(x^*) \leq f_{1q}^{UAM}(x)$ *for at least* $f_{1q}^{UAM}(x)$.

Definition 3. x^* *is a surly Pareto optimal solution if and only if* x^* *is the Pareto optimal solution of the PRBL-MOFPP for the UAM and* $x^* \in S_*(\gamma^0)$. *Otherwise, this solution is called a possibly Pareto optimal solution.*

3. The surely and possibly SSFK for PRBL-MOFPP

Now, the primary inquiry is: once the UAM and LAM of the PRBL-MOFP have been solved, to what extent may its data regarding δ and γ be altered without compromising the efficiency of its certainly and possibly Pareto optimal solution?

As a result, the following is the definition of the set of feasible parameters, the solvability set, and SSFK for PRBL-MOFPP:

Definition 4. *The set of feasible parameters for the UAM and LAM of PRBL-MOFPP, respectively, is defined by [26,47]:*

$$V^{UAM} = \{\delta \in R^T | S^*(x, \delta) \neq \emptyset\}. \qquad V^{LAM} = \{\gamma \in R^t | S_*(x, \gamma^0) \neq \emptyset\} \tag{48}$$

Definition 5. *The solvability set for the UAM and LAM of PRBL-MOFPP, respectively, is denoted by:*

$$B^{UAM} = \{\delta \in R^T | x^* \text{ is a possily pareto optimal solution for UAM}\}$$

$$B^{LAM} = \{\gamma \in R^t | x^* \text{ is a surly pareto optimal solution for LAM}\}$$

For any $\gamma \in \mathrm{B}^{\mathrm{LAM}}, \delta \in \mathrm{B}^{\mathrm{UAM}}$, if there is a surely or possibly Pareto optimal solution, then the surely and possibly SSFK can be defined.

Definition 6. *Suppose that* $\gamma = \gamma^0$, $\delta = \delta^0$ *where* $S_*(\gamma^0) \subseteq S \subseteq S^*(\delta^0)$ *with a pareto optimal solution* $\boldsymbol{x}^*$ *for problem* $(42) - (47)$, *then the surely SSFK for the PRBL-MOFPP denoted by* $S^L(\boldsymbol{x}^*, \gamma, D_{rq}^-, D_{rq}^+)$ *is defined by:*

$$S^L(\boldsymbol{x}^*, \gamma, D_{rq}^-, D_{rq}^+) = \{\gamma \in R^t | \boldsymbol{x}^* \text{ is an optimal solution for problem } (42) - (47)\}$$

Definition 7. *Suppose that* $\gamma = \gamma^0, \delta = \delta^0$ *where* $S_*(\gamma^0) \subseteq S \subseteq S^*(\delta^0)$ *with a pareto optimal solution* $\boldsymbol{x}^*$ *for problem* $(36) - (41)$, *then the possibly SSFK for the PRBL-MOFPP denoted by* $S^U(\boldsymbol{x}^*, \delta, D_{rq}^-, D_{rq}^+)$ *is defined by:*

$$S^U(\boldsymbol{x}^*, \delta, D_{rq}^-, D_{rq}^+) = \{\delta \in R^T | \boldsymbol{x}^* \text{ is an optimal solution for problem } (36) - (41)\}$$

The surely and possibly SSFK for PRBL-MOFPP is the set of all parameters corresponding to one surely or possibly Pareto optimal solution [7,47,51,52]. It is simple to observe that the PRBL-MOFPP is stable. (7)–(10) implies the stability of the final parametric FGP model for the UAM:

$$min\ Z_{rq}^{UAM} = \sum_{q=1}^{Q_1} w_{1q}^- d_{1q}^- + \sum_{q=1}^{Q_2} w_{2q}^- d_{2q}^- \tag{49}$$

subject to

$$\left[L_{rq} c_{rq}^T - L_{rq}^0 d_{rq}^T\right] x + D_{rq}^- - D_{rq}^+ = \left[L_{rq}^0 \beta_{rq} - L_{rq} \alpha_{rq}\right] \quad r = 1,2; q = 1, \dots, Q_r \tag{50}$$

$$\sum_{j=1}^{n} A_{rj} x_j \leq \delta_r, \qquad r = 1,2 \tag{51}$$

$$-d_{rq}^T x + D_{rq}^- \leq \beta_{rq}, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{52}$$

$$x_{1v}^{UAM} = x_{1v}^{UAM^*} \qquad v = 1,2, \dots, n_1 \tag{53}$$

$$x_{rv} \geq 0, \qquad (r = 1,2) \quad v = 1,2, \dots, n_r \tag{54}$$

$$D_{rq}^-, D_{rq}^+ \geq 0, \quad D_{rq}^- \times D_{rq}^+ = 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{55}$$

Similarly, the final parametric FGP model for the LAM:

$$min\ Z_{rq}^{LAM} = \sum_{q=1}^{Q_1} w_{1q}^- d_{1q}^- + \sum_{q=1}^{Q_2} w_{2q}^- d_{2q}^- \tag{56}$$

subject to

$$\left[L_{rq} c_{rq}^T - L_{rq}^0 d_{rq}^T\right] x + D_{rq}^- - D_{rq}^+ = \left[L_{rq}^0 \beta_{rq} - L_{rq} \alpha_{rq}\right] \quad r = 1,2; q = 1, \dots, Q_r \tag{57}$$

$$\sum_{j=1}^{n} a_{rj} x_j \leq \gamma_r, \qquad r = 1,2 \tag{58}$$

$$-d_{rq}^T x + D_{rq}^- \leq \beta_{rq}, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{59}$$

$$x_{1v}^{LAM} = x_{1v}^{LAM^*} \qquad v = 1,2, \dots, n_1 \tag{60}$$

$$x_{rv} \geq 0, \qquad (r = 1,2) \quad v = 1,2, \dots, n_r \tag{61}$$

$$D_{rq}^-, D_{rq}^+ \geq 0, \quad D_{rq}^- \times D_{rq}^+ = 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{62}$$

Employment of the KKT optimality for PRBL-MOFPP

The Lagrangian function for UAM of PRBL-MOFPP (49)–(55) follows as [36,38,47]:

$$L = \left[\sum_{q=1}^{Q_1} w_{1q}^{-} D_{1q}^{-} + \sum_{q=1}^{Q_2} w_{2q}^{-} D_{2q}^{-}\right]$$
$$+\lambda_{rq}\left[\left[L_{rq}c_{rq}^{T} - L_{rq}^{0}d_{rq}^{T}\right]x + D_{rq}^{-} - D_{rq}^{+} - \left[L_{rq}^{0}\beta_{rq} - L_{rq}\alpha_{rq}\right]\right] + \xi_{1v}\left[x_{1v}^{UAM} - x_{1v}^{*UAM}\right]$$
$$-\psi_{rv}x_{rv} + \mu_{rq}\left[-d_{rq}^{T}x + D_{rq}^{-} - \beta_{rq}\right] + \eta_r\left[\sum_{r=1}^{2} A_{rj}\,x_j - \delta_r\right] \tag{63}$$
$$+\varphi_{rq}\left[-D_{rq}^{-}\right] + \vartheta_{rq}\left[-D_{rq}^{+}\right]$$

where λ, ξ, ψ, μ, η, φ and ϑ are the Lagrange multipliers. Then the KKT optimality conditions [16,19,31,48–50] for the UAM of the PRBL-MOFPP (49)–(55), will have the following form:

$$\frac{\partial L}{\partial x_j} = \lambda_{rq}\left[L_{rq}c_{rq}^{T} - L_{rq}^{0}d_{rq}^{T}\right] + \xi_{1v} - \psi_{rv} - \mu_{rq}d_{rq}^{T} - \sum_{r=1}^{2}\eta_r A_{rj} = 0, (j = 1,2 \dots, n) \tag{64}$$

$$\frac{\partial L}{\partial D_{rq}^{-}} = w_{rq}^{-} + \lambda_{rq} + \mu_{rq} - \varphi_{rq} = 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{65}$$

$$\frac{\partial L}{\partial D_{rq}^{+}} = -\lambda_{rq} - \vartheta_{rq} = 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{66}$$

$$\left[L_{rq}c_{rq}^{T} - L_{rq}^{0}d_{rq}^{T}\right]x + D_{rq}^{-} - D_{rq}^{+} - \left[L_{rq}^{0}\beta_{rq} - L_{rq}\alpha_{rq}\right] = 0, \qquad \forall r, q \tag{67}$$

$$x_{1v}^{UAM} - x_{1v}^{UAM^*} = 0, \qquad v = 1,2, \dots, n_1 \tag{68}$$

$$-d_{rq}^{T}x + D_{rq}^{-} - \beta_{rq} \le 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{69}$$

$$\sum_{r=1}^{2} A_{rj}\,x_j - \delta_r \le 0, \qquad (j = 1,2 \dots, n) \tag{70}$$

$$D_{rq}^{-}, D_{rq}^{+} \ge 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{71}$$

$$x_{rv} \ge 0, \qquad (r = 1,2), \left(v = 1,2, \dots, n_r\right) \tag{72}$$

$$\mu_{rq}\left[-d_{rq}^{T}x + D_{rq}^{-} - \beta_{rq}\right] = 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{73}$$

$$\eta_r\left[\sum_{r=1}^{2} A_{rj}\,x_j - \delta_r\right] = 0 \qquad (r = 1,2) \tag{74}$$

$$\varphi_{rq}\left[-D_{rq}^{-}\right] = 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{75}$$

$$\vartheta_{rq}\left[-D_{rq}^{+}\right] = 0, \qquad (r = 1,2), (q = 1,2, \dots, Q_r) \tag{76}$$

$$\psi_{rv}x_{rv} = 0, \qquad (r = 1,2), (v = 1,2, \dots, n_r) \tag{77}$$

$$\psi, \mu, \eta, \varphi, \vartheta \ge 0, \qquad \text{and} \qquad \lambda, \xi \in R \tag{78}$$

where all the KKT condition phrases (64)–(78) are examined at a Pareto optimal solution $\boldsymbol{x}^{*UAM}$ of the FGP model. Solving the system of Equations (64)–(78), the surly or possibly SSFK for PRBL-MOFPP will be obtained.

4. Algorithm for investigating the SSFK for PRBL-MOFPP

The talk above will be followed by the creation of an algorithm for obtaining the surly or possibly SSFK for PRBL-MOFPP as:

Algorithm 1 investigating the SSFK for PRBL-MOFPP	
$\boldsymbol{phase\ I}$: $Obtain\ a\ compromise\ solution\ of\ PRBL-MOFPP$	
Step 1.	Put $\gamma=\gamma^0, \delta=\delta^0\ as\ S_*(\gamma^0)\subseteq S\subseteq S^*(\delta^0)$.
Step 2.	Compute $u_{rq}^0,\ g_{rq}^0, w_{rq}^-,\ r=1,2;\ q=1,2,\dots,Q_r$.
Step 3.	Formulate the membership functions $\mu_{rq}\left(f_{rq}(x)\right) q=1,2,\dots,Q_r$, as in Equation (16).
Step 4.	Do the linearization process for $\mu_{rq}\left(f_{rq}(x)\right) q=1,2,\dots,Q_r$ using Equations (36)–(38).
Step 5.	Solve the FLDM FGP model to get $x_{1v}^{UAM}=x_{1v}^{*UAM}$.
Step 6.	Formulate and solve the FGP model, as in Equations (39)–(44). to get a compromise solution $\boldsymbol{x}^{*\boldsymbol{UAM}}$.
Step 7.	If $x^{*UAM}\in S_*(\gamma^0)$ go to Step 8, otherwise go to phase II to get $S^U(x^*,\delta,D_{rq}^-,D_{rq}^+)$.
Step 8.	Formulate and Solve the LAM by executing: Steps 2 to Step 6, then go to phase II Step 11, to get $S^L(\boldsymbol{x}^*,\gamma,D_{rq}^-,D_{rq}^+)$.
$\boldsymbol{phase\ II}$: $Determination\ of\ surly\ or\ possibly\ SSFK$	
Step 9.	Formulate the Lagrangian function, for the FGP of UAM, Equations (67)–(81).
Step 10.	Apply the KKT conditions to get $\boldsymbol{S^U}(\boldsymbol{x}^*,\boldsymbol{\delta},\boldsymbol{D_{rq}^-},\boldsymbol{D_{rq}^+})$. Go to Step 12.
Step 11.	Execute: Step 9 to Step10 for LAM, to get $\boldsymbol{S^L}(\boldsymbol{x}^*,\boldsymbol{\gamma},\boldsymbol{D_{rq}^-},\boldsymbol{D_{rq}^+})$.
Step 12.	Stop.

5. Illustrative numerical example

Consider the following PRBL-MOFPP with parameters on the right-hand side of the rough feasible region.

5.1. Case 1

[Upper Level]

$$\underbrace{max}_{x_1}\left(f_{11}(x)=\frac{2x_1+5x_2}{x_1+x_2+8},\qquad f_{12}(x)=\frac{2x_1+x_2}{x_1+3x_2+1}\right)$$

where x_2 solves

[Lower Level]

$$\underbrace{max}_{x_2}\left(f_{21}(x)=\frac{3x_1+x_2-1}{3x_1+5x_2+2},\qquad f_{22}(x)=\frac{4x_1+x_2+2}{x_1+x_2+6}\right)$$

subject to

$$\mathrm{x}\in S,\quad \text{where}\quad S_*(\gamma)\subseteq S\subseteq S^*(\delta)$$

$$S^*(\delta) = \left\{ (x_1, x_2) \in R^2 \middle| \begin{array}{l} x_1 + 3x_2 \le \delta_1, \\ 6x_1 + 7x_2 \le \delta_2, \\ x_1, x_2 \ge 0. \end{array} \right\}, \quad S_*(\gamma) = \left\{ (x_1, x_2) \in R^2 \middle| \begin{array}{l} x_1 + x_2 \le \gamma_1, \\ x_1 + 4x_2 \le \gamma_2, \\ x_1, x_2 \ge 0. \end{array} \right\}$$

Let $\delta_1 = 21, \delta_2 = 60, \gamma_1 = 1, \gamma_2 = 2$.

Formulate and solve the for the UAM as:

[Upper Level]

$$\underbrace{max}_{x_1} \left(f_{11}(x) = \frac{2x_1 + 5x_2}{x_1 + x_2 + 8}, \qquad f_{12}(x) = \frac{2x_1 + x_2}{x_1 + 3x_2 + 1} \right)$$

where x_2 solves

[Lower Level]

$$\underbrace{max}_{x_2} \left(f_{21}(x) = \frac{3x_1 + x_2 - 1}{3x_1 + 5x_2 + 2}, \qquad f_{22}(x) = \frac{4x_1 + x_2 + 2}{x_1 + x_2 + 6} \right)$$

subject to

$$x \in S^*(\delta^0) = \left\{ (x_1, x_2) \in R^2 \middle| \begin{array}{l} x_1 + 3x_2 \le 21, \\ 6x_1 + 7x_2 \le 60, \\ x_1, x_2 \ge 0. \end{array} \right\}$$

Table 1 summarizes each of the maximum and minimum values. The determined aspiration levels, upper tolerance limits, and weights w_{rq}^0 are also included.

Table 1. individual maximum, minimum values, u_{rq}^0, g_{rq}^0 and weights w_{rq}^0.

	$\boldsymbol{f_{11}(x)}$	$\boldsymbol{f_{12}(x)}$	$\boldsymbol{f_{21}(x)}$	$\boldsymbol{f_{22}(x)}$
$max\ (f_{rq}(x))$	2.33333	1.818182	0.90625	2.625
$min\ (f_{rq}(x))$	0	0	−0.5	0.33333
u_{rq}^0	2.3	1.8	0.9	2.6
g_{rq}^0	0	0	−0.5	0.3
w_{rq}^-	0.43	0.55	0.71	0.43

Table 2 displays the linearized membership coefficients.

Table 2. The coefficient of the linearized membership goals $\left(\boldsymbol{C}_{rq}\right)^T$ and $\boldsymbol{G}_{\boldsymbol{rq}}$.

	$\boldsymbol{f_{11}(x)}$	$\boldsymbol{f_{12}(x)}$	$\boldsymbol{f_{21}(x)}$	$\boldsymbol{f_{22}(x)}$
$\left(\boldsymbol{C}_{rq}\right)^T$	$\begin{pmatrix} -0.14 \\ 1.15 \end{pmatrix}^T$	$\begin{pmatrix} 0.1 \\ -2.45 \end{pmatrix}^T$	$\begin{pmatrix} 0.195 \\ -2.515 \end{pmatrix}^T$	$\begin{pmatrix} 0.591 \\ -0.699 \end{pmatrix}^T$
G_{rq}	8	1	7.42	5.914

Solving FGP model for FLDM:

$$min\,Z = 0.43D_{11}^- + 0.55D_{12}^-$$

subject to

$$-0.14x_1 + 1.15x_2 + D_{11}^- - D_{11}^+ = 8$$

$$0.1x_1 - 2.45x_2 + D_{12}^- - D_{12}^+ = 1$$

$$-x_1 - x_2 + D_{11}^- \le 8, \qquad -x_1 - 3x_2 + D_{12}^- \le 1$$

$$x_1 + 3x_2 \le 21, \qquad 6x_1 + 7x_2 \le 60$$

$$x_1, x_2, D_{11}^-, D_{11}^+, D_{12}^-, D_{12}^+ \ge 0$$

Using Lingo 20 programming software, the satisfactory solution of the FLDM is obtained as $(x_1^*, x_2^*) = (0,0)$.

Solving FGP model for PRBL-MOFPP:

$$min\, Z = 1.099D_{11}^- + 0.877D_{12}^- + 1.587D_{21}^- + 0.386D_{22}^-$$

subject to

$$-0.14x_1 + 1.15x_2 + D_{11}^- - D_{11}^+ = 8$$

$$0.1x_1 - 2.45x_2 + D_{12}^- - D_{12}^+ = 1$$

$$0.195x_1 - 2.515x_2 + D_{21}^- - D_{21}^+ = 7.42$$

$$0.591x_1 - 0.699x_2 + D_{22}^- - D_{22}^+ = 5.914$$

$$-x_1 - x_2 + D_{11}^- \le 8, \qquad -x_1 - 3x_2 + D_{12}^- \le 1$$

$$-3x_1 - 5x_2 + D_{21}^- \le 2, \qquad -x_1 - x_2 + D_{22}^- \le 6$$

$$x_1 + 3x_2 \le 21, \qquad 6x_1 + 7x_2 \le 60$$

$$x_1 = 0, \qquad x_2, D_{11}^-, D_{11}^+, D_{12}^-, D_{12}^+, D_{21}^-, D_{21}^+, D_{22}^-, D_{22}^+ \ge 0$$

Using Lingo 20 programming software, the satisfactory solution, thus $(x_1^*, x_2^*) = (0,1.94614)$. And, $(D_{11}^-, D_{12}^-,, D_{21}^-, D_{22}^-, D_{11}^+, D_{12}^+, D_{21}^+, D_{22}^+)$ = (5.761939,5.768043, 11.73070,7.274352,0,0,0,0), which is a possibly Pareto optimal solution for the PRBL-MOFPP introduced in example 1, so, $S^U(x^*, \delta, D_{rq}^{*-}, D_{rq}^{*+})$ is determined. The stability of UAM, implies the stability of the following parametric FGP:

$$min\ Z = 1.099D_{11}^- + 0.877D_{12}^- + 1.587D_{21}^- + 0.386D_{22}^-$$

subject to

$$-0.14x_1 + 1.15x_2 + D_{11}^- - D_{11}^+ = 8$$

$$0.1x_1 - 2.45x_2 + D_{12}^- - D_{12}^+ = 1$$

$$0.195x_1 - 2.515x_2 + D_{21}^- - D_{21}^+ = 7.42$$

$$0.591x_1 - 0.699x_2 + D_{22}^- - D_{22}^+ = 5.914$$

$$-x_1 - x_2 + D_{11}^- \le 8, \qquad -x_1 - 3x_2 + D_{12}^- \le 1$$

$$-3x_1 - 5x_2 + D_{21}^- \le 2, \qquad -x_1 - x_2 + D_{22}^- \le 6$$

$$x_1 + 3x_2 \le \delta_1$$

$$6x_1 + 7x_2 \le \delta_2$$

$$x_1 = 0, \ \ x_2, D_{11}^-, D_{11}^+, D_{12}^-, D_{12}^+, D_{21}^-, D_{21}^+, D_{22}^-, D_{22}^+ \ge 0$$

The Lagrangian function of the above problem is formulated as:

$$L = 1.099D_{11}^{-} + 0.877D_{12}^{-} + 1.587D_{21}^{-} + 0.386D_{22}^{-}$$
$$+\lambda_1[-0.14x_1 + 1.15x_2 + D_{11}^{-} - D_{11}^{+} - 8] + \lambda_2[0.1x_1 - 2.45x_2 + D_{12}^{-} - D_{12}^{+} - 1]$$
$$+\lambda_3[0.195x_1 - 2.515x_2 + D_{21}^{-} - D_{21}^{+} - 7.42]$$
$$+\lambda_4[0.591x_1 - 0.699x_2 + D_{22}^{-} - D_{22}^{+} - 5.914] + \xi[x_1] + \mu_1[-x_1 - x_2 + D_{11}^{-} - 8]$$
$$+\mu_2[-x_1 - 3x_2 + D_{12}^{-} - 1] + \mu_3[-3x_1 - 5x_2 + D_{21}^{-} - 2] + \mu_4[-x_1 - x_2 + D_{22}^{-} - 6]$$
$$+\eta_1[x_1 + 3x_2 - \delta_1] + \eta_2[6x_1 + 7x_2 - \delta_2] + \psi[-x_2] + \phi_1[-D_{11}^{-}] + \varphi_1[-D_{11}^{+}]$$
$$+\phi_2[-D_{12}^{-}] + \varphi_2[-D_{12}^{+}] + \phi_3[-D_{21}^{-}] + \varphi_3[-D_{21}^{+}] + \phi_4[-D_{22}^{-}] + \varphi_4[-D_{22}^{+}]$$

where $\psi, \mu, \eta, \varphi, \vartheta \geq 0$ and $\lambda, \xi \in R$, the KKT conditions for the optimal solution to the parametric UAM:

$$\frac{\partial L}{\partial x_1} = -0.14\lambda_1 + 0.1\lambda_2 + 0.195\lambda_3 + 0.591\lambda_4 + \xi - \mu_1 - \mu_2 - 3\mu_3 - \mu_4 + \eta_1 + 6\eta_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 1.15\lambda_1 - 2.45\lambda_2 - 2.515\lambda_3 - 0.699\lambda_4 - \mu_1 - 3\mu_2 - 5\mu_3 - \mu_4 + 3\eta_1 + 7\eta_2 - \psi = 0$$

$$\frac{\partial L}{\partial D_{11}^{-}} = 0.43 + \lambda_1 + \mu_1 - \phi_1 = 0$$

$$\frac{\partial L}{\partial D_{11}^{+}} = -\lambda_1 - \varphi_1 = 0$$

$$\frac{\partial L}{\partial D_{12}^{-}} = 0.55 + \lambda_2 + \mu_2 - \phi_2 = 0$$

$$\frac{\partial L}{\partial D_{12}^{+}} = -\lambda_2 - \varphi_2 = 0$$

$$\frac{\partial L}{\partial D_{21}^{-}} = 0.71 + \lambda_3 + \mu_3 - \phi_3 = 0$$

$$\frac{\partial L}{\partial D_{21}^{+}} = -\lambda_3 - \varphi_3 = 0$$

$$\frac{\partial L}{\partial D_{22}^{-}} = 0.43 + \lambda_4 + \mu_4 - \phi_4 = 0$$

$$\frac{\partial L}{\partial D_{22}^{+}} = -\lambda_4 - \varphi_4 = 0$$

$$\mu_1[-x_1 - x_2 + D_{11}^{-} - 8] = 0$$

$$\mu_2[-x_1 - 3x_2 + D_{12}^{-} - 1] = 0$$

$$\mu_3[-3x_1 - 5x_2 + D_{21}^{-} - 2] = 0$$

$$\mu_4[-x_1 - x_2 + D_{22}^{-} - 6] = 0$$

$$\eta_1[x_1 + 3x_2 - \delta_1] = 0$$

$$\eta_2[6x_1 + 7x_2 - \delta_2] = 0$$

$$\psi[-x_2] = 0$$

$$\phi_1[-D_{11}^{-}] = 0$$

$$\phi_2[-D_{12}^-] = 0$$
$$\phi_3[-D_{21}^-] = 0$$
$$\phi_4[-D_{22}^-] = 0$$
$$\varphi_1[-D_{11}^+] = 0$$
$$\varphi_2[-D_{12}^+] = 0$$
$$\varphi_3[-D_{21}^+] = 0$$
$$\varphi_4[-D_{22}^+] = 0$$

Solving the previous system, we obtain: $\mu_1 = \mu_2 = \mu_4 = \psi = \phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$, and $\mu_3, \phi_1, \phi_2, \phi_3, \phi_4 > 0$. Therefore, the possibly SSFK is given by:

$$S^U(0,1.94164) = \left\{ (\delta_1, \delta_2) \in R^2 \middle| \begin{array}{l} 5.83842\eta_1 - \eta_1\delta_1 = 0, 13.622\eta_2 - \eta_2\delta_2 = 0, \\ \delta_1 \geq 5.82492, \delta_2 \geq 13.59148, \\ 2.32\lambda_3 + 8\mu_3 - 4\eta_1 - 8\eta_2 - \xi = 0.90464, \\ \eta_1, \eta_2 > 0, \lambda_3 < -0.71. \end{array} \right\}$$

5.2. Case 2

[Upper Level]

$$\underbrace{max}_{x_1} \left(f_{11}(x) = \frac{2x_1 + 5x_2}{x_1 + x_2 + 8}, \qquad f_{12}(x) = \frac{2x_1 + x_2}{x_1 + 3x_2 + 1} \right)$$

where x_2 solves

[Lower Level]

$$\underbrace{max}_{x_2} \left(f_{21}(x) = \frac{3x_1 + x_2 - 1}{3x_1 + 5x_2 + 2}, \qquad f_{22}(x) = \frac{4x_1 + x_2 + 2}{x_1 + x_2 + 6} \right)$$

subject to

$$x \in S \text{ where } \qquad S_*(\gamma) \subseteq S \subseteq S^*(\delta)$$

$$S^*(\delta) = \left\{ (x_1, x_2) \in R^2 \middle| \begin{array}{l} x_1 + 3x_2 \leq \delta_1, \\ 6x_1 + 7x_2 \leq \delta_2, \\ x_1, x_2 \geq 0. \end{array} \right\}, S_*(\gamma) = \left\{ (x_1, x_2) \in R^2 \middle| \begin{array}{l} x_1 + x_2 \leq \gamma_1, \\ x_1 + 4x_2 \leq \gamma_2, \\ x_1, x_2 \geq 0. \end{array} \right\}$$

Let $\delta_1 = 21, \delta_2 = 60, \gamma_1 = 7, \gamma_2 = 24$.

Initially, resolve the issue for the upper approximation set. From example 1, the optimal solution of the FGP for the upper approximation set is obtained as $(x_1^0, x_2^0) = (0,1.94614)$, which is a surely Pareto optimal solution for example 2, so the surely stability set of the first kind will be investigated. So, the following LAM will be solved:

[Upper Level]

$$\underbrace{max}_{x_1} \left(f_{11}(x) = \frac{2x_1 + 5x_2}{x_1 + x_2 + 8}, \qquad f_{12}(x) = \frac{2x_1 + x_2}{x_1 + 3x_2 + 1} \right)$$

where x_2 solves

[Lower Level]

$$\underbrace{max}_{x_2}\left(f_{21}(x)=\frac{3x_1+x_2-1}{3x_1+5x_2+2}, \qquad f_{22}(x)=\frac{4x_1+x_2+2}{x_1+x_2+6}\right)$$

subject to

$$x_1+x_2\le 7,$$
$$x_1+4x_2\le 24,$$
$$x_1,x_2\ge 0$$

Table 3 summarizes each of the maximum and minimum values. The determined aspiration levels, upper tolerance limits, and weights w_{rq}^0 are also included.

Table 3. Individual maximum, minimum values, u_{rq}^0, g_{rq}^0 and w_{rq}^0.

	$\boldsymbol{f_{11}(x)}$	$\boldsymbol{f_{12}(x)}$	$\boldsymbol{f_{21}(x)}$	$\boldsymbol{f_{22}(x)}$
$max\ (f_{rq}(x))$	2.33333	1.714286	0.85	2.166667
$min\ (f_{rq}(x))$	0	0	−0.5	0.33333
u_{rq}^0	2.3	1.7	0.8	2.1
g_{rq}^0	0	0	−0.5	0.3
w_{rq}^0	0.435	0.588	0.769	0.555

Table 4 shows the coefficients of the linearized membership goals.

Table 4. The coefficient of the linearized membership goals $\left(G_{rq}^0\right)^T$ and G_{rq}^0.

	$\boldsymbol{f_{11}(x)}$	$\boldsymbol{f_{12}(x)}$	$\boldsymbol{f_{21}(x)}$	$\boldsymbol{f_{22}(x)}$
$\left(G_{rq}^0\right)^T$	$\begin{pmatrix}-0.13\\1.175\end{pmatrix}^T$	$\begin{pmatrix}0.176\\-2.412\end{pmatrix}^T$	$\begin{pmatrix}0.4605\\-2.3085\end{pmatrix}^T$	$\begin{pmatrix}1.054\\-0.611\end{pmatrix}^T$
G_{rq}^0	8	1	2	5.886

Solving the FLDM FGP model:

$$min\,Z=0.435D_{11}^-+0.588D_{12}^-$$

subject to

$$-0.13x_1+1.175x_2+D_{11}^--D_{11}^+=8$$

$$0.176x_1-2.412x_2+D_{12}^--D_{12}^+=1$$

$$-x_1-x_2+D_{11}^-\le 8$$

$$-x_1-3x_2+D_{12}^-\le 1$$

$$x_1+x_2\le 7$$

$$4x_1+3x_2\le 24$$

$$x_1,x_2,D_{11}^-,D_{11}^+,D_{12}^-,D_{12}^+\ge 0$$

Using Lingo 20 programming software, the satisfactory solution of the FLDM is obtained as $(x_1^0,x_2^0)=(5.681818,0)$.

Solving the FGP model of BL-MOFPP:

$$min\,Z=0.435D_{11}^-+0.588D_{12}^-+0.769D_{21}^-+0.555D_{22}^-$$

subject to

$$-0.13x_1 + 1.175x_2 + D_{11}^{-} - D_{11}^{+} = 8$$

$$0.176x_1 - 2.412x_2 + D_{12}^{-} - D_{12}^{+} = 1$$

$$0.4605x_1 - 2.3085x_2 + D_{21}^{-} - D_{21}^{+} = 2$$

$$1.054x_1 - 0.611x_2 + D_{22}^{-} - D_{22}^{+} = 5.886$$

$$-x_1 - x_2 + D_{11}^{-} \leq 8$$

$$-x_1 - 3x_2 + D_{12}^{-} \leq 1$$

$$-3x_1 - 5x_2 + D_{21}^{-} \leq 2$$

$$-x_1 - x_2 + D_{22}^{-} \leq 6$$

$$x_1 + x_2 \leq 7$$

$$4x_1 + 3x_2 \leq 24$$

$$x_1 = 5.681818$$

$$x_2, D_{11}^{-}, D_{11}^{+}, D_{12}^{-}, D_{12}^{+}, D_{21}^{-}, D_{21}^{+}, D_{22}^{-}, D_{22}^{+} \geq 0$$

Using Lingo 20 programming software, the satisfactory solution of the SLDM is obtained as: $(x_1^0, x_2^0) = (5.681818, 0)$ and $(D_{11}^{-}, D_{12}^{-},, D_{21}^{-}, D_{22}^{-}, D_{11}^{+}, D_{12}^{+}, D_{21}^{+}, D_{22}^{+}) = (8.738636, 0.00000032, 0, 0, 0, 0, 0.6164772, 0.1526363)$. The stability of LAM implies the stability of the next parametric FGP:

$$min\, Z = 0.435D_{11}^{-} + 0.588D_{12}^{-} + 0.769D_{21}^{-} + 0.555D_{22}^{-}$$

subject to

$$-0.13x_1 + 1.175x_2 + D_{11}^{-} - D_{11}^{+} = 8$$

$$0.176x_1 - 2.412x_2 + D_{12}^{-} - D_{12}^{+} = 1$$

$$0.4605x_1 - 2.3085x_2 + D_{21}^{-} - D_{21}^{+} = 2$$

$$1.054x_1 - 0.611x_2 + D_{22}^{-} - D_{22}^{+} = 5.886$$

$$-x_1 - x_2 + D_{11}^{-} \leq 8$$

$$-x_1 - 3x_2 + D_{12}^{-} \leq 1$$

$$-3x_1 - 5x_2 + D_{21}^{-} \leq 2$$

$$-x_1 - x_2 + D_{22}^{-} \leq 6$$

$$x_1 + x_2 \leq \gamma_1$$

$$4x_1 + 3x_2 \leq \gamma_2$$

$$x_1 = 5.681818$$

$$x_2, D_{11}^{-}, D_{11}^{+}, D_{12}^{-}, D_{12}^{+}, D_{21}^{-}, D_{21}^{+}, D_{22}^{-}, D_{22}^{+} \geq 0$$

Therefore, the Lagrangean function is formulated as:

$$
\begin{aligned}
L = & \ 0.435D_{11}^{-} + 0.588D_{12}^{-} + 0.769D_{21}^{-} + 0.555D_{22}^{-} \\
& +\lambda_1[-0.13x_1 + 1.175x_2 + D_{11}^{-} - D_{11}^{+} - 8] \\
& +\lambda_2[0.176x_1 - 2.412x_2 + D_{12}^{-} - D_{12}^{+} - 1] \\
& +\lambda_3[0.4605x_1 - 2.3085x_2 + D_{21}^{-} - D_{21}^{+} - 2] \\
& +\lambda_4[1.054x_1 - 0.611x_2 + D_{22}^{-} - D_{22}^{+} = 5.886] \\
& +\xi[x_1 - 5.681818] + \mu_1[-x_1 - x_2 + D_{11}^{-} - 8] \\
& +\mu_2[-x_1 - 3x_2 + D_{12}^{-} - 1] + \mu_3[-3x_1 - 5x_2 + D_{21}^{-} - 2] + \mu_4[-x_1 - x_2 + D_{22}^{-} - 6] \\
& +\eta_1[x_1 + x_2 - \gamma_1] + \eta_2[4x_1 + 3x_2 - \delta_2] + \psi[-x_2] + \phi_1[-D_{11}^{-}] + \varphi_1[-D_{11}^{+}] \\
& +\phi_2[-D_{12}^{-}] + \varphi_2[-D_{12}^{+}] + \phi_3[-D_{13}^{-}] + \varphi_3[-D_{21}^{+}] + \phi_4[-D_{22}^{-}] + \varphi_4[-D_{22}^{+}]
\end{aligned}
$$

where $\psi, \mu, \eta, \varphi, \vartheta \geq 0$, and $\lambda, \xi \in R$, the KKT conditions for the optimal solution to the parametric LAM:

$$\frac{\partial L}{\partial x_1} = 0.13\lambda_1 + 0.176\lambda_2 + 0.4605\lambda_3 + 1.054\lambda_4 + \xi - \mu_1 - \mu_2 - 3\mu_3 - \mu_4 + \eta_1 + 4\eta_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 1.175\lambda_1 - 2.412\lambda_2 - 2.3085\lambda_3 - 0.61\lambda_4 - \mu_1 - 3\mu_2 - 5\mu_3 - \mu_4 + \eta_1 + 3\eta_2 - \psi = 0$$

$$\frac{\partial L}{\partial D_{11}^{-}} = 0.435 + \lambda_1 + \mu_1 - \phi_1 = 0$$

$$\frac{\partial L}{\partial D_{11}^{+}} = -\lambda_1 - \varphi_1 = 0$$

$$\frac{\partial L}{\partial D_{12}^{-}} = 0.588 + \lambda_2 + \mu_2 - \phi_2 = 0$$

$$\frac{\partial L}{\partial D_{12}^{+}} = -\lambda_2 - \varphi_2 = 0$$

$$\frac{\partial L}{\partial D_{21}^{-}} = 0769 + \lambda_3 + \mu_3 - \phi_3 = 0$$

$$\frac{\partial L}{\partial D_{21}^{+}} = -\lambda_3 - \varphi_3 = 0$$

$$\frac{\partial L}{\partial D_{22}^{-}} = 0.555 + \lambda_4 + \mu_4 - \phi_4 = 0$$

$$\frac{\partial L}{\partial D_{22}^{+}} = -\lambda_4 - \varphi_4 = 0$$

$$\mu_1[-x_1 - x_2 + D_{11}^{-} - 8] = 0$$

$$\mu_2[-x_1 - 3x_2 + D_{12}^{-} - 1] = 0$$

$$\mu_3[-3x_1 - 5x_2 + D_{21}^{-} - 2] = 0$$

$$\mu_4[-x_1 - x_2 + D_{22}^{-} - 6] = 0$$

$$\eta_1[x_1 + x_2 - \gamma_1] = 0$$

$$\eta_2[4x_1 + 3x_2 - \gamma_2] = 0$$

$$\psi[-x_2] = 0$$

$$\phi_1[-D_{11}^-] = 0$$
$$\varphi_1[-D_{11}^+] = 0$$
$$\phi_2[-D_{12}^-] = 0$$
$$\varphi_2[-D_{12}^+] = 0$$
$$\phi_3[-D_{21}^-] = 0$$
$$\varphi_3[-D_{21}^+] = 0$$
$$\phi_4[-D_{22}^-] = 0$$
$$\varphi_4[-D_{22}^+] = 0$$

and, $\lambda_2 = -1.1976$, $\lambda_3 = -2.0747$, $\lambda_4 = 0$, and $\xi \in R$ also, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \phi_1 = \phi_2 = \varphi_3 = \varphi_4 = 0$, and $\psi, \varphi_1, \varphi_2, \phi_4 > 0$. Therefore, the surely SSFK is given by.

$$S^L(5.681818,0) = \left\{ (\gamma_1, \gamma_2) \in R^2 \middle| \begin{array}{l} 2\eta_1 + 7\eta_2 + \xi - \psi = -1.376511, \\ \gamma_1 \geq 5.681818, \gamma_2 \geq 22.7273, \\ 5.681818\eta_1 - \eta_1\gamma_1 = 0, \\ 22.7273\eta_1 - \eta_2\gamma_2 = 0 \\ \xi \in R, \eta_1, \eta_2, \psi > 0 \end{array} \right\}$$

6. Conclusion

In this paper we present the PRBL-MOFPP, in which the parameters exist on the right-hand side of the rough set of constraints. For such a model, the FGP is employed to solve both the UAM and LAM. Moreover, the Lagrangian function and the KKT optimality conditions are presented. Finally, the surely and possibly SSFK is defined. Illustrative examples were given to clarify the applicability and efficiency of the proposed model.

However, there are numerous other topics that should be researched and discussed in the future in parametric ML-OP in rough environment, such as:

1) Parametric multi-level multi-objective fractional programming problems in rough environments.
2) Parametric multi-level multi-objective quadratic programming problems in rough environments.

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