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Article

Petrov-Galerkin Lucas polynomials approach for solving the time-fractional diffusion equation

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Abstract: In this research paper, a spectral method is used for numerically solving the time-fractional diffusion equation as the time fractional diffusion equations are a powerful tool for simulating physical systems. We employ the Lucas polynomials (LPs) with Petrov-Galerkin for the linear combination basis. The main idea of the proposed technique is to convert the governed boundary-value problem into a system of linear algebraic equations by applying the Petrov-Galerkin method. Many procedures can solve the resulting linear system. The method's accuracy is shown through several examples.

Keywords: time-fractional diffusion equation; Lucas polynomials; spectral methods

1. Introduction

Several fields within the applied sciences rely heavily on fractional differential equations (FDEs). Conventional differential equations fail to account for a great deal of the events that they represent. This is because of their remarkable capacity to simulate intricate inheritance and memory processes. For instance, as mentioned by Magin [1], this approach mimics a wide range of biological and physiological processes, such as the growth of tumors and the actions of neurons. The aforementioned equations can also be used to explain anomalous diffusion, electromagnetic phenomena, and wave propagation in complex media [2]. FDEs have also frequently been used to depict the intricate mechanical response of viscoelastic materials to stress or strain [3]. Fractional calculus has been utilized in signal processing for feature extraction, denoising, and filtering [4].

Particularly in numerical solutions for ordinary, partial, and differential equations, spectral approaches are crucial to numerical analysis. Using spectral methods, the differential equation solution is expressed as a sum of basis functions, and the coefficients are then selected to minimize the error between the exact and numerical solutions. Spectral approaches have the advantage of rapidly converting differential problems into linear or nonlinear algebraic equation solutions [5,6]. The collocation, tau, and Galerkin methods are the three widely used spectral approaches. Numerous researchers have given these techniques a lot of thought [7–10]. The equation for fractional diffusion treats super-diffusive flow phenomena and generalizes the classical diffusion equation, making it one of the foundational equations of mathematical physics. In recent years, its demand has grown. The old version of the diffusion process issue, which attempts to derive past field conditions from present data, has a wide range of applications. The well-known simple model of the time-diffusion problem is the familiar partial differential equation of the temperature field $u(x,t)$ [11,12]

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, t\in \mathbb{R}^+, x\in \sigma \subset \mathbb{R}^1, \quad (1)$$

under a predetermined beginning state at t=0 and specific boundary constraints on ∂σ. Finding a closed form of $u(x, t)$ using only the beginning and boundary conditions provided is the goal. Time fractional diffusion equations have many applications such as description reaction diffusion processes with memory effects in biology, physics and chemistry. Modeling heat conduction with memory effects in material is one of the important applications of time fractional diffusion equations. The view and use of Lucas polynomials in contemporary research are highly interesting, and they play important roles in mathematical theory and practice. Many scholars have examined their diverse mathematical characteristics and come up with a number of important conclusions. For instance, Koundal [13] developed new formulae on shifted LPs, Abd-Elhameed et al. [14] developed new formulae on Fibonacci and LPs, Gumgum et al. [15] suggested a Lucas polynomials collocation approach to solve functional integro-differential equations, Singh and Ray [16] used a spectral Lucas approach to solve multi-dimensional stochastic Itô-Volterra integral equations, and Youssri et al. [17] presented a generalized Lucas Galerkin method for solving the linear one_dimensional telegraph type equation.

We use LPs with Petrov-Galerkin for solving time fractional diffusion equation because less calculation is required, and the resulting errors are small. It gives us high accuracy and efficiency.

The overall organization of the article is as follows: In Section 2, Caputo fractional calculus along with LPs are discussed in depth along with their fundamental relations. In Section 3 we introduce Petrov-Galerkin Approach for the Treatment of Time Fractional diffusion equation. In Section 4 we introduce some examples to confirm the accuracy of our method, and we make a comparison between our method and other. Finally, we present the conclusion for our method in Section 5.

2. Preliminaries and fundamentals

The first part of this section, we recall some definitions and properties of fractional calculus. The second part we recall some properties and relations of LPs.

2.1. Some definitions and properties of the fractional Calculus

Definition 1. [18] *On the standard Lebesgue space* $L_1[0,1]$, *the Riemann-Liouville fractional integral operator* I^ρ *of order* ρ *is defined as:*

$$I^\rho h(y)=\begin{cases}\frac{1}{\Gamma(\rho)}\int_0^y (y-t)^{\rho-1}h(t)dt, & \rho>0,\\ h(y), & \rho=0.\end{cases} \quad (2)$$

Definition 2. [18,19] *The fractional-order derivative is defined by Caputo as follows:*

$$D^\rho h(y)=\frac{1}{\Gamma(m-\rho)}\int_0^y (y-t)^{m-\rho-1}h^{(m)}(t)dt,\ \ \rho>0,\ \ y>0, \quad (3)$$

where $m-1 \leqslant \rho < m,\ m \in \mathbb{N}$. *The operator* D^{ρ} *satisfies the following properties for* $m-1 \leqslant \rho < m,\ m \in \mathbb{N}$,

$$(i)\ (D^{\rho}\, I^{\rho} h)(y) = h(y),$$

$$(ii)\ (I^{\rho}\, D^{\rho} h)(y) = h(y) - \sum_{k=0}^{m-1} \frac{h^{(k)}(0^{+})}{\Gamma(k+1)} (y-a)^{k}, \quad y > 0,$$

$$(iii) D^{\rho}\, y^{k} = \frac{\Gamma(k+1)}{\Gamma(k+1-\rho)}\, y^{k-\rho}, \quad k \in \mathbb{N}, \quad k \geq \lceil \rho \rceil,$$

where $\lceil \rho \rceil$ *indicates the smallest integer greater than or equal to* ρ.

Definition 3. [18]

$$D^{\nu} x^{k} = \begin{cases} 0, & if \quad k \in \mathbb{N}_0 \text{ and } k < \lceil \nu \rceil, \\ \dfrac{\Gamma(k+1)}{\Gamma(k+1-\nu)} x^{k-\nu}, & if \quad k \in \mathbb{N}_0 \text{ and } k \geq \lceil \nu \rceil, \end{cases} \tag{4}$$

where $\mathbb{N} = \{1,2,\dots\}$ *and* $\mathbb{N}_0 = \{0,1,2,\dots\}$.

2.2. Some properties and fundamental relations of LPs

The recurrence relation governing LPs $L_i(\mathrm{x})$ is defined in Youssri et al. [17]

$$L_i(x) = x L_{i-1}(x) + L_{i-2}(x), \qquad i \geq 2, \tag{5}$$

where

$$L_0(x) = 2, L_1(x) = x, \tag{6}$$

The analytic form of $L_i(x)$ is

$$L_i(x) = i \sum_{r=0}^{\lfloor \frac{i}{2} \rfloor} \frac{\binom{i-r}{r}}{i-r} x^{i-2r}, \qquad i \geq 1, \tag{7}$$

or, in another form

$$L_i(\mathrm{x}) = 2i \sum_{k=0}^{i} \frac{\binom{\frac{i+k}{2}}{\frac{i-k}{2}} \delta_{i+k}}{i+k} \mathrm{x}^{k}, \qquad i \geq 1, \tag{8}$$

where

$$\delta_r = \begin{cases} 1, & if \quad r \quad even, \\ 0, & if \quad r \quad odd. \end{cases} \tag{9}$$

The Binet's form for $L_i(\mathrm{x})$ can be expressed by the following form

$$L_i(x) = \frac{(x + \sqrt{x^2+4})^{i} + (x - \sqrt{x^2+4})^{i}}{2^{i}}, \qquad i \geq 0, \tag{10}$$

It is important to note that the following Fibonacci recurrence relation

$$L_{i+2} = L_{i+1} + L_i, \quad L_0 = 2, L_1 = 1,$$

or LPs can be used to construct the well-known Lucas numbers L_i by setting $x = 1$.

3. Petrov-Galerkin technique for the treatment of time-fractional diffusion equation

Consider the following time-fractional diffusion equation (TFDE):

$$D_t^{\alpha} u(\mathrm{x},t) - \beta u_{\mathrm{xx}}(\mathrm{x},t) = f(\mathrm{x},t), \qquad 0 < \alpha \leq 1, \tag{11}$$

given the initial and boundary conditions

$$u(\mathrm{x},0) = \sigma(\mathrm{x}), \quad 0 < \mathrm{x} \leq 1, \tag{12}$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t \leq 1, \tag{13}$$

where $h(\mathrm{x},t)$ is the known source term and β is arbitrary known positive constant.

3.1. Trial functions

Assuming the following basis functions

$$\psi_i(\mathrm{x}) = \mathrm{x} \int_{\mathrm{x}}^{1} L_i(t) dt, \tag{14}$$

$$\phi_j(t) = L_j(t). \tag{15}$$

3.2. Petrov-Galerkin solution for TFDE

Now, assuming the following spaces functions

$$S_M = span\{\psi_i(x)\phi_j(t) i,j = 0,1,\ldots,M\}, \tag{16}$$

$$V_M = \{u \in S_M, u(0,t) = u(1,t) = 0\}, \tag{17}$$

then, any function $u(\mathrm{x},t) \in V_M$ may be written as

$$u_M(x,t) = \sum_{i=0}^{M} \sum_{j=0}^{M} c_{ij} \psi_i(x) \phi_j(t). \tag{18}$$

Now, the application of Petrov-Galerkin is used to find $u_M(\mathrm{x},t) \in V_M$ such that

$$\left((D_t^{\alpha} u_M(\mathrm{x},t), \mathrm{x}^r t^s)\right) - \beta\left((u_{Mxx}(\mathrm{x},t), \mathrm{x}^r t^s)\right) = \left((f(\mathrm{x},t), \mathrm{x}^r t^s)\right), \ \ 0 \leq r \leq M, 0 \leq s \leq M-1, \tag{19}$$

where,

$$((u(x,t), v(x,t))) = \int_0^1 \int_0^1 (u(x,t) v(x,t) dx dt. \tag{20}$$

Now, for $0 \leq r \leq M, 0 \leq s \leq M-1$, Equation (19) can be written as

$$\sum_{i=0}^{M} \sum_{i=0}^{M} c_{ij} (\psi_i(x), x^r) \left(D_t^{\alpha} \phi_j(t), t^s\right) - \beta \sum_{i=0}^{M} \sum_{i=0}^{M} c_{ij} (\psi_i''(x), x^r) \left(\phi_j(t), t^s\right) = \left((f(x,t), x^r t^s)\right), \tag{21}$$

or in the simplest form

$$\sum_{i=0}^{M}\sum_{i=0}^{M} c_{ij} g_{i,r} b_{j,s} - \beta \sum_{i=0}^{M}\sum_{i=0}^{M} c_{ij} d_{i,r} h_{j,s} = f_{r,s}, 0 \leq r \leq M, 0 \leq s \leq M-1, \tag{22}$$

along with the following initial condition

$$\sum_{i=0}^{M}\sum_{i=0}^{M} c_{ij} \psi_i \left(\frac{k+1}{M+2}\right) \phi_j(0) = \sigma\left(\frac{k+1}{M+2}\right), k = 0, \ldots, M, \tag{23}$$

where

$$b_{j,s} = \left(D_t^{\alpha} \phi_j(t), t^s\right), d_{i,r} = (\psi_i''(\mathrm{x}), \mathrm{x}^r), g_{i,r} = (\psi_i(\mathrm{x}), \mathrm{x}^r), h_{j,s} = \left(\phi_j(t), t^s\right), f_{r,s} = \left((f(\mathrm{x},t), \mathrm{x}^r t^s)\right). \tag{24}$$

Therefore, a linear system of algebraic equations of dimension (M + 1) × (M + 1) in the unknown expansion coefficients c_{ij} is produced by Equations (22) and (23) and can be solved using an appropriate approach.

Theorem 1. *The elements $b_{j,s}, d_{i,r}, g_{i,r}$ and $h_{j,s}$ are given by*

$$b_{j,s} = \sum_{k=1}^{j} \frac{(2j)k! \begin{pmatrix} \frac{j+k}{2} \\ \frac{j-k}{2} \end{pmatrix} \delta_{j+k}}{(j+k)\Gamma(k-\alpha-1)(k-\alpha+s+1)}, \tag{25}$$

$$d_{i,r} = -2i \sum_{k=0}^{i} \begin{pmatrix} \frac{i+k}{2} \\ \frac{i-k}{2} \end{pmatrix} \delta_{i+k} \frac{(1+k)(2+k)}{(1+k+r)(i+k)}, \tag{26}$$

$$g_{i,r} = 2i \sum_{k=0}^{i} \begin{pmatrix} \frac{i+k}{2} \\ \frac{i-k}{2} \end{pmatrix} \delta_{i+k} \frac{1}{(2+r)(3+k+r)(i+k)}, \tag{27}$$

$$h_{j,s} = \frac{1}{j+s+1} \; {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}, -\frac{j}{2}-\frac{s}{2}-\frac{1}{2} \\ 1-j, -\frac{j}{2}-\frac{s}{2}+\frac{1}{2} \end{matrix} \right| -4 \right), \tag{28}$$

where $\_rF_s$ indicates the Gauss generalized hypergeometric function defined by

$${}_rF_s\left(\left. \begin{matrix} p_1, p_2, \cdots, p_r \\ q_1, q_2, \cdots, q_s \end{matrix} \right| \mathrm{t} \right) = \Sigma_{m=0}^{\infty} \frac{(p_1)_m \, (p_2)_m \cdots (p_r)_m}{(q_1)_m \, (q_2)_m \cdots (qs)_m} \frac{t^m}{m!}. \tag{29}$$

Proof. The elements of $b_{j,s}, h_{j,s}$ exist in Reference [20].

To find $d_{i,r} = (\psi_i'', \mathrm{x}^r)$, we have

$$\psi_i(x) = x \int_x^1 L_i(t) dt, \tag{30}$$

$$\psi_i(\mathrm{x}) = \mathrm{x}\int_{\mathrm{x}}^{1} L_i(t)dt$$
$$= \mathrm{x}\int_{\mathrm{x}}^{1} 2i\sum_{k=0}^{i}\frac{\begin{pmatrix}\frac{i+k}{2}\\ \frac{i-k}{2}\end{pmatrix}\delta_{i+k}}{i+k}t^k dt = 2i\sum_{k=0}^{i}\frac{\begin{pmatrix}\frac{i+k}{2}\\ \frac{i-k}{2}\end{pmatrix}\delta_{i+k}}{(i+k)(k+1)}\mathrm{x}(1-\mathrm{x}^{k+1}), \tag{31}$$

so we get,

$$\psi_i''(\mathrm{x}) = 2i\sum_{k=0}^{i}\frac{\begin{pmatrix}\frac{i+k}{2}\\ \frac{i-k}{2}\end{pmatrix}\delta_{i+k}}{(i+k)}(1-\mathrm{x})^{-1+k}(-2+(2+k)\mathrm{x}), \tag{32}$$

then,

$$d_{i,r} = (\psi_i(\mathrm{x})'', \mathrm{x}^r) = 2i\sum_{k=0}^{i}\frac{\begin{pmatrix}\frac{i+k}{2}\\ \frac{i-k}{2}\end{pmatrix}\delta_{i+k}}{(i+k)}\int_0^1 (1-\mathrm{x})^{-1+k}(-2+(2+k)\mathrm{x})\mathrm{x}^r dx$$
$$= -2i\sum_{k=0}^{i}\begin{pmatrix}\frac{i+k}{2}\\ \frac{i-k}{2}\end{pmatrix}\delta_{i+k}\frac{(1+k)(2+k)}{(1+k+r)(i+k)}. \tag{33}$$

To find $g_{i,r} = (\psi_i(\mathrm{x}), \mathrm{x}^r)$.

$$g_{i,r} = (\psi_i(\mathrm{x}), \mathrm{x}^r)$$
$$= 2i\sum_{k=0}^{i}\frac{\begin{pmatrix}\frac{i+k}{2}\\ \frac{i-k}{2}\end{pmatrix}\delta_{i+k}}{(i+k)(k+1)}\int_0^1 \mathrm{x}(1-\mathrm{x}^{k+1})\,\mathrm{x}^r d\mathrm{x}$$
$$= 2i\sum_{k=0}^{i}\begin{pmatrix}\frac{i+k}{2}\\ \frac{i-k}{2}\end{pmatrix}\delta_{i+k}\frac{1}{(2+r)(3+k+r)(i+k)}.\ \square \tag{34}$$

4. Illustrative examples

Test Problem 1. [21] *Consider the following TFDE*

$$D_t^{\alpha}u(\mathrm{x},t) - \frac{\partial^2 u(\mathrm{x},t)}{\partial \mathrm{x}^2} = f(\mathrm{x},t), \quad (\mathrm{x},t) \in (0,1)\times(0,1), \tag{35}$$

given boundary conditions

$$u(0,t) = 0, u(1,t) = 0 \quad 0 \le \mathrm{x} \le 1, \tag{36}$$

and initial condition

$$u(\mathrm{x},0) = 0, \tag{37}$$

where $f(x,t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}\sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x)$, *the exact solution of this example is* $u(x,t) = t^2\sin(2\pi x)$.

The approximate spectral solution (left) and exact solution (right) at $\gamma = 0.5$ and $M = 7$ are displayed in **Figure 1**. The L_∞ error for $\gamma = 0.5$ and $M = 7$ is displayed in **Figure 2**. The absolute error (AE) at $\gamma = 0.5$ and $M = 7$ at various t values is displayed in **Table 1**. The maximum absolute error (MAE) at various γ and M values is displayed in **Table 2**.

Table 1. The AE of Test Problem 1.

***γ* =0.5**			
x	***t* = 0.1**	***t* = 0.5**	***t* = 0.9**
0.1	2.104×10^{-6}	5.857×10^{-6}	1.901×10^{-5}
0.2	1.677×10^{-6}	4.660×10^{-6}	1.510×10^{-5}
0.3	2.861×10^{-6}	7.970×10^{-6}	2.588×10^{-5}
0.4	1.901×10^{-6}	5.305×10^{-6}	1.726×10^{-5}
0.5	2.758×10^{-12}	3.544×10^{-12}	6.111×10^{-12}
0.6	1.901×10^{-6}	5.305×10^{-6}	1.726×10^{-5}
0.7	2.861×10^{-6}	7.970×10^{-6}	2.588×10^{-5}
0.8	1.677×10^{-6}	4.660×10^{-6}	1.510×10^{-5}
0.9	2.104×10^{-6}	5.857×10^{-6}	1.901×10^{-5}

Table 2. MAE of Test Problem 1.

M	$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 0.9$
1	4.169×10^{-1}	4.120×10^{-1}	4.043×10^{-1}
3	7.749×10^{-2}	7.693×10^{-2}	7.634×10^{-2}
5	2.508×10^{-3}	7.693×10^{-2}	2.473×10^{-3}
7	3.203×10^{-5}	3.197×10^{-5}	3.191×10^{-5}

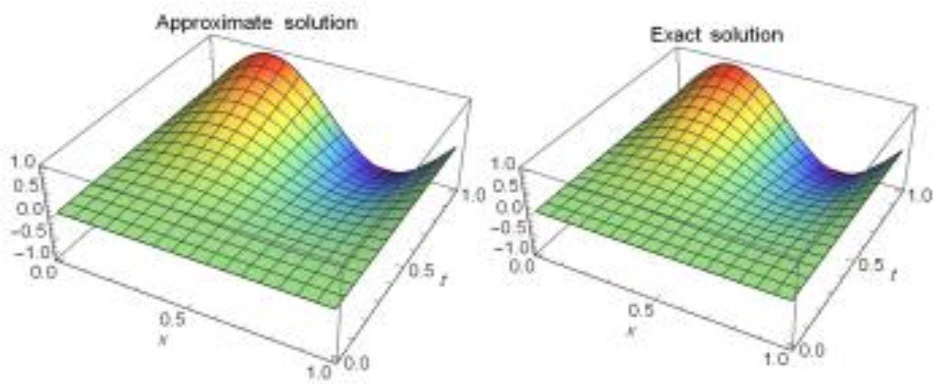

Figure 1. The approximate spectral solution (left) and exact solution (right) of Test Problem 1.

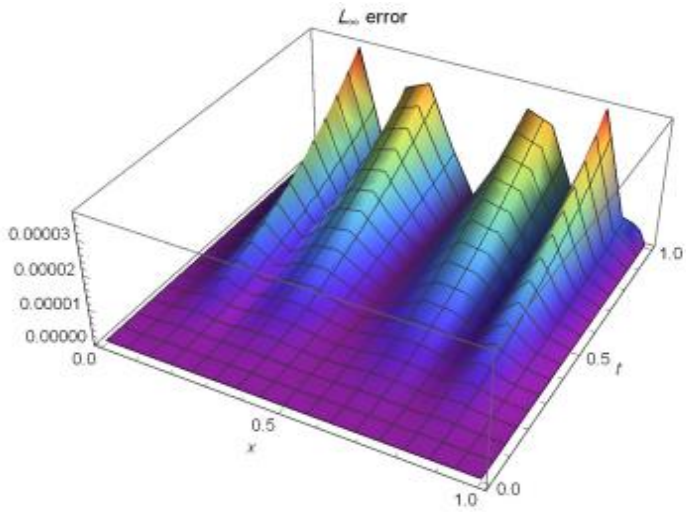

Figure 2. L_∞ error of Test Problem 1.

Test Problem 2. [21] *Consider the following TFDE*

$$D_t^\alpha u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad (x,t) \in (0,1) \times (0,1), \tag{38}$$

given boundary conditions

$$u(0,t) = 0, u(1,t) = 0 \quad 0 \le x \le 1, \tag{39}$$

and initial condition

$$u(x,0) = 0, \tag{40}$$

The exact solution of this example is $u(x,t) = \sin(\pi x)\sin(\pi t)$.

Figure 3 displays the approximate solution (left) and exact solution (right) at $\gamma = 0.5$ and $M = 8$. **Figure 4** shows the L_∞ error at $\gamma = 0.5$ and $M = 8$. **Table 3** shows the AE at $\gamma = 0.5$ and $M = 8$ at different values of t. **Table 4** shows the MAE at distinct values of γ and M. Finally, **Table 5** presents a comparison of the AE at $\gamma = 0.5$ between our method and method in [21].

Table 3. The AE of Test Problem 2.

	$\boldsymbol{\gamma = 0.5}$		
X	***t* =0.1**	***t* =0.5**	***t* =0.9**
0.1	5.628×10^{-9}	3.627×10^{-10}	2.436×10^{-8}
0.2	1.275×10^{-8}	1.842×10^{-9}	4.556×10^{-8}
0.3	1.769×10^{-8}	2.714×10^{-9}	6.265×10^{-8}
0.4	1.951×10^{-8}	1.592×10^{-9}	7.416×10^{-8}
0.5	2.03×10^{-8}	1.487×10^{-9}	7.803×10^{-8}
0.6	1.9516×10^{-8}	1.595×10^{-9}	7.416×10^{-8}
0.7	1.771×10^{-8}	2.720×10^{-9}	6.265×10^{-8}
0.8	1.276×10^{-8}	1.849×10^{-9}	4.556×10^{-8}
0.9	5.634×10^{-9}	3.572×10^{-10}	2.436×10^{-8}

Table 4. MAE of Test Problem 2.

$\boldsymbol{M}$	$\boldsymbol{\gamma = 0.1}$	$\boldsymbol{\gamma = 0.5}$	$\boldsymbol{\gamma = 0.9}$
2	6.839×10^{-2}	6.925×10^{-2}	6.552×10^{-2}
4	1.326×10^{-3}	1.490×10^{-3}	1.706×10^{-3}
6	1.269×10^{-5}	8.516×10^{-6}	1.950×10^{-5}
8	8.814×10^{-8}	7.803×10^{-8}	9.745×10^{-8}

Table 5. Comparison between our technique and the technique by Roul et al. [21] for Test Problem 2.

$\boldsymbol{\gamma = 0.5}$	
Method in [21] at M = 32 and Δx = 0.001	Our method at M = 8
1.10×10^{-3}	7.803×10^{-8}

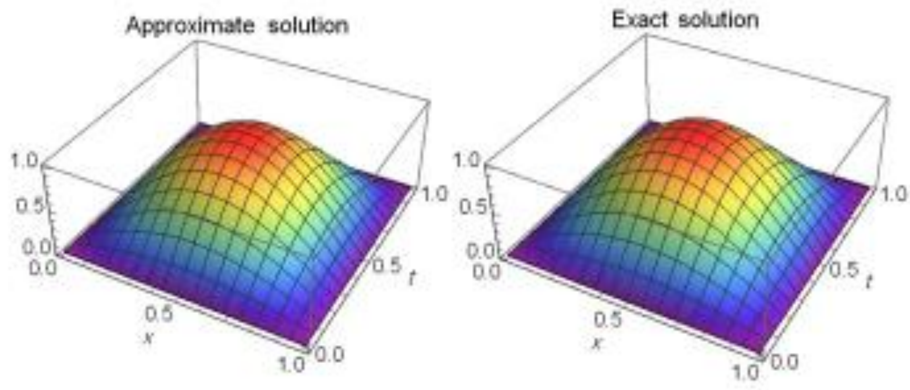

Figure 3. The approximate solution (left) and exact solution (right) of Test Problem 2.

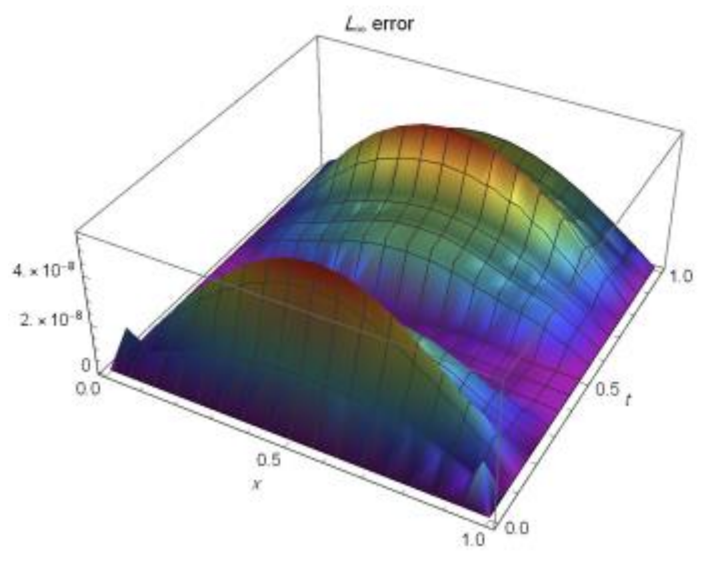

Figure 4. L_∞ error of Test Problem 2.

Test Problem 3. *Consider the following TFDE*

$$D_t^{\alpha} u(\mathrm{x}, t) - \frac{\partial^2 u(\mathrm{x}, t)}{\partial \mathrm{x}^2} = f(\mathrm{x}, t), \quad (\mathrm{x}, t) \in (0,1) \times (0,1), \tag{41}$$

boundary conditions

$$u(0, t) = 0, u(1, t) = 0 \quad 0 \leq x \leq 1, \tag{42}$$

and initial condition

$$u(\mathrm{x}, 0) = 0. \tag{43}$$

The exact solution of this problem is $u(x, t) = t^2(1 - x)\sin(x)$.

The approximate spectral solution (left) and exact smooth solution (right) are displayed in **Figure 5** at $\gamma = 0.5$ and $M = 8$. The L_∞ error at$\gamma = 0.5$ and $M = 8$ is displayed in **Figure 6**. **Table 6** illustrates the AE at $\gamma = 0.5$ and $M = 8$ at different values of t. The MAE at different values of γ and N is displayed in **Table 7**.

Table 6. The AE of Test Problem 3.

	$\boldsymbol{\gamma = 0.5}$		
$\boldsymbol{x}$	$\boldsymbol{t}$ **= 0.1**	$\boldsymbol{t}$ **= 0.5**	$\boldsymbol{t}$ **= 0.9**
0.1	3.506×10^{-14}	1.577×10^{-14}	4.955×10^{-14}
0.2	3.119×10^{-14}	5.954×10^{-14}	3.718×10^{-13}
0.3	4.778×10^{-14}	6.193×10^{-14}	4.372×10^{-13}
0.4	9.288×10^{-14}	3.072×10^{-14}	1.749×10^{-13}
0.5	6.896×10^{-14}	5.188×10^{-14}	4.601×10^{-13}
0.6	4.058×10^{-14}	1.289×10^{-13}	7.072×10^{-13}
0.7	7.373×10^{-14}	1.486×10^{-14}	3.108×10^{-13}
0.8	6.326×10^{-14}	1.054×10^{-15}	2.052×10^{-13}
0.9	1.681×10^{-14}	5.616×10^{-14}	3.048×10^{-13}

Table 7. MAE of Test Problem 3.

$\boldsymbol{M}$	$\boldsymbol{\gamma = 0.1}$	$\boldsymbol{\gamma = 0.5}$	$\boldsymbol{\gamma = 0.9}$
2	3.135×10^{-4}	3.099×10^{-4}	3.061×10^{-4}
4	5.508×10^{-7}	5.469×10^{-7}	5.428×10^{-7}
6	4.914×10^{-10}	4.912×10^{-10}	4.910×10^{-10}
8	8.594×10^{-13}	9.264×10^{-13}	4.367×10^{-13}

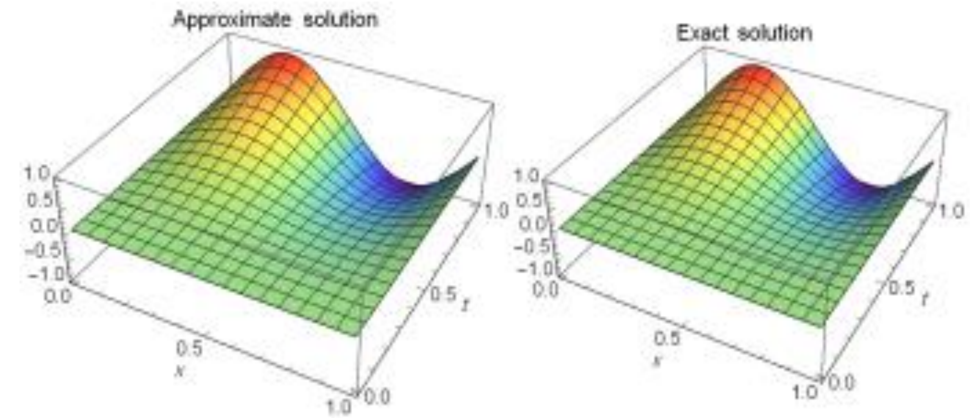

Figure 5. The approximate spectral solution (left) and exact smooth solution (right) of Test Problem 3.

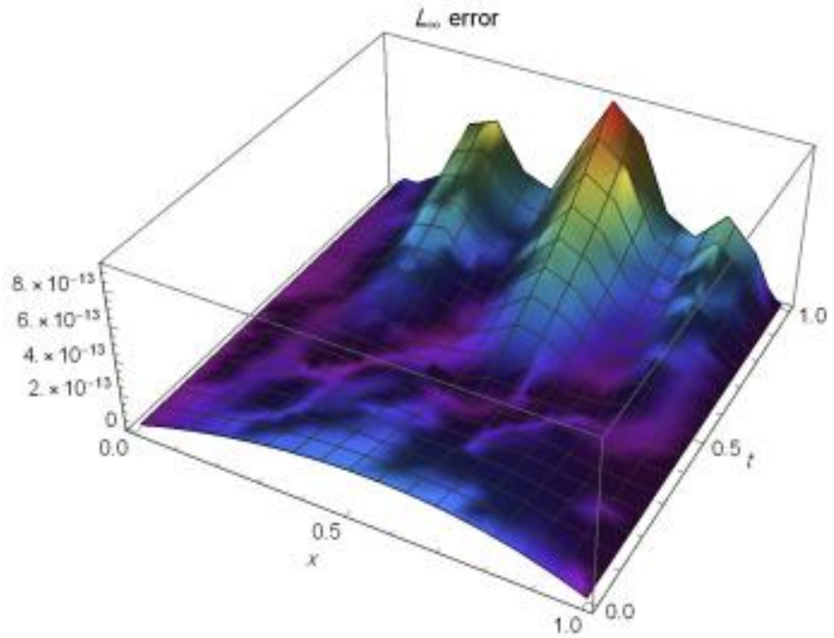

Figure 6. L_∞error of Test Problem 3.

5. Conclusion

A numerical petrov-Galerkin method for resolving the time_fractional diffusion problem was introduced in this work. Appropriate sets of basis functions were selected using LPs and their modified polynomials. The proposed Petrov-Galerkin algorithm is based on the idea that a problem should be reduced to a system that can be solved with the help of proper solver. The numerical findings showed that when expressed as combinations of our modified basis function. Also, the approximate answers agree with the precise ones. The resulting errors are small. It gives us high accuracy and efficiency. As an expected future work, we aim to solve other problems as in Ahmed [22] and Zaky et al. [23]. All codes were written and debugged by Mathematica 12 on Dell Inspiron 15, Processor: Intel (R) Core(TM) i5-5200U CPU $@$ 2.20 GHz 2.20GHz, 8GB Ram DDR3 and 1024 GB storage. Finally, readers who wish to use the presented algorithm can follow the steps outlined in Algorithm 1.

Algorithm 1 Coding algorithm for the proposed technique in Subsection 3.2

1: Input
α, β, $\sigma(x)$ and $f(x,t)$
2: Step 1
Let that the approximate solution $u_M(\mathrm{x}, t)$ as in (18)
3: Step 2
Applying the Petrov-Galerkin method to get the system in (22) and (24)
4: Step 3
Employing Theorem 1 to obtain the elements $b_{j,s}, d_{i,r}, g_{i,r}$ *and* $h_{j,s}$
5: Step 4
Use NSolve command to solve the system in (22) and (24) to get c_{ij}.
6: Output
$u_M(\mathrm{x}, t)$

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