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Commentary

On "A new three-step fixed point iteration scheme with strong convergence and applications"

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Abstract: This note delves into the convergence analysis of several iterative methods and elucidates their behaviors. Furthermore, we demonstrate that the findings presented in "A new three-step fixed point iteration scheme with strong convergence and applications" are not entirely novel. In particular, some of the results either overlap with or restate previously established methods without introducing significant innovations.

Keywords: algorithms; iterative methods; convergence order

Introduction and preliminaries

Let D be a convex subset of a normed space E and $T : D \to D$ be a selfmap. Let p be the fixed point of T.

(a) Considering the Picard iteration scheme

$$x_{n+1} = Tx_n,\ n = 0, 1, 2 \dots \tag{1}$$

starting with a suitable initial approximation x_0, we built up a sequence of approximations, say $\{x_n\}$, for the fixed point p of T.

For the scalar case, $D = [a, b] \subset \mathbb{R}$.The scheme will be converge to p, provided that

(i) the initial approximation x_0 is chosen in the interval $[a, b]$,

(ii) $|T^{'}x| < 1$ for all $x \in [a, b]$,

(iii) $a \le Tx \le b$ for all $x \in [a, b]$.

Definition 1. *Let $\{x_n\}$ converges to ν. If there exist an integer ρ and a real positive constant C such that*

$$\lim_{n\to\infty} \frac{|x_{n+1} - \nu|}{(x_n - \nu)^{\rho}} = C$$

then ρ is called the order of convergence. The efficiency index of an iterative method is a metric used to compare different iterative methods. It is defined as EI=$\rho^{\frac{1}{\lambda}}$, where ρ is the local order of convergence of the method and λ is the number of function evaluations needed to carry out the method per iteration [1].

To determine the order of convergence of the sequence $\{x_n\}$, let us consider the Taylor expansion of Tx_n

$$Tx_n = Tx + \frac{T^{'}x}{1!}(x_n - x) + \frac{T^{''}x}{2!}(x_n - x)^2 + \dots + \frac{T^{k}x}{k!}(x_n - x)^k + \dots$$

We have

$$x_{n+1} - x = \frac{T^{'}x}{1!}(x_n - x) + \frac{T^{''}x}{2!}(x_n - x)^2 + ... + \frac{T^k x}{k!}(x_n - x)^k + ...$$

Theorem 1. *Suppose that $T \in C^n[a, b]$. If $T^k x = 0$, for $k = 1, 2, ..., \rho - 1$ and $T^k x \neq 0$, then the sequence $\{x_n\}$ has ρ as its order of convergence* [1].

Remark 1. *(1). It is well known that the fixed point method has first order of convergence.*

(2). The well known multistep Picard method is given by:

$$\begin{array}{rcl} x_0 & \in & D, \\ y_n^{(0)} & = & x_n, \\ y_n^{(k)} & = & Ty_n^{(k-1)}, \\ x_{n+1} & = & Ty_n^{(k)},\ n = 0, 1, 2, ..., k = 0, 1, 2, ... \end{array} \tag{2}$$

and has the linear order of convergence [1].

The following algorithm is due to Kang et al. [2]:

For a given $x_0 \epsilon [a, b]$, we calculate the approximation solution x_{n+1}, by the iteration scheme

$$\begin{array}{rcl} x_{n+1} & = & \frac{\theta x_n + Tx_n}{1+\theta} = \frac{-T^{'}x_n x_n + Tx_n}{1-T^{'}x_n}, \\ \theta & = & -T^{'}x_n, n = 0, 1, 2, ... \end{array} \tag{3}$$

Remark 2. *(1). The value of θ arises from the derivation steps for algorithm (3). It plays a crucial role in the convergence analysis and behavior of the algorithm. Specifically, θ is a parameter that influences the iterative process and helps determine the efficiency and accuracy of the method. Its exact value and influence are derived from the underlying mathematical formulation and assumptions in the iterative scheme.*

(2). For the implementation of algorithm (3), it is advisable to adhere to the conventional convergence criteria of fixed-point methods, as outlined in [2].

The following algorithm is due to Sharma P., et al. ((11) of [3]):

$$\begin{array}{rcl} x_0 & \in & [a, b], \\ z_n & = & \frac{mx_n + Tx_n}{1+m}, \\ y_n & = & Tz_n, \\ x_{n+1} & = & Ty_n, m > 0 \text{ is a real number}, n = 0, 1, 2, ... \end{array} \tag{4}$$

which is the combination of (3) with the two-step composition of fixed point method.

(b) For arbitrary $x_0 \in D$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \lambda_n)\, x_n + \lambda_n T x_n,\ n \geq 0 \tag{5}$$

where $\lambda_n \in [0, 1]$, is known as the Mann iteration scheme [4] .

(c) For arbitrary $x_0 \in D$, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = (1-\lambda_n)\, x_n + \lambda_n T y_n, \\ y_n = (1-\lambda_n')\, x_n + \lambda_n' T x_n,\ n \geq 0 \end{cases} \tag{6}$$

where $\lambda_n, \lambda_n' \in [0,1]$, is known as the Ishikawa iteration scheme [5].

(d) For arbitrary $x_0 \in D$, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = (1-\lambda_n)\, x_n + \lambda_n T y_n, \\ y_n = (1-\lambda_n')\, x_n + \lambda_n' T z_n, \\ z_n = (1-\lambda_n'')\, x_n + \lambda_n'' T x_n,\ n \geq 0 \end{cases} \tag{7}$$

where $\lambda_n, \lambda_n', \lambda_n'' \in [0,1]$, is known as the Noor iteration scheme [6].

We comment as follows:

(1). The algorithm (7) in [3] and the predictor-step of algorithm (4) are not novel; they are actually identical to algorithm (3) proposed by Kang et al. [2].

(2). The order of convergence of algorithm (3) is two, requiring two function evaluations per iteration and yielding an efficiency index of $2^{\frac{1}{2}}$, as stated in [2]. Therefore, the results presented in Theorem 2 ([27] of [3]) coincide with the established proof of convergence for algorithm (3) under the condition $m = -T^{'}x_n$. Consequently, Theorem 2 in [3] does not introduce any new findings and is essentially a restatement of previously known results from [2].

(3). The algorithm (4) has the order of convergence two for $m = -T^{'}x_n$, as stated in [3], requiring four evaluations per iteration with efficiency index $2^{\frac{1}{4}} < 2^{\frac{1}{2}}$.

(4). In algorithm (4), the claim about $m > 0$ is not always true as $m = -T^{'}x_n$.

(5). From algorithm (3), we have

$$\begin{aligned} x_{n+1} &= \tfrac{\theta x_n + T x_n}{1+\theta} \\ &= \tfrac{\theta}{1+\theta} x_n + \tfrac{1}{1+\theta} T x_n \\ &= (1-\lambda) x_n + \lambda T x_n;\ \lambda = \tfrac{1}{1+\theta} \in (0,1] \end{aligned} \tag{8}$$

which is the well known Kirik-Mann type algorithm [4,7] for $\lambda_n = \lambda = \frac{1}{1+\theta} \in (0,1]$. It is now evident that algorithm (3) and the Kirik-Mann type algorithm [4,7] are equivalent, thus sharing the same order of convergence under specific conditions.

(6). The algorithm (4) assumes the following form:

$$\begin{aligned} x_0 &\in [a,b], \\ z_n &= (1-\lambda) x_n + \lambda T x_n;\ \lambda = \tfrac{1}{1+m} \in (0,1], \\ y_n &= T z_n, \\ x_{n+1} &= T y_n,\ n = 0,1,2,... \end{aligned} \tag{9}$$

It follows that (3) and (4) exhibit identical convergence orders under certain conditions. Furthermore, algorithm (4) essentially represents the special case of three-step iteration method due to Noor [6] for $\lambda''_n = \lambda = \frac{1}{1+m} \in (0,1]$, $\lambda'_n = 1$ and $\lambda_n = 1$, rendering algorithm (5) in [3] non-novel. Moreover, algorithms (5) in [3] and (4) are equivalent, thus rendering Theorem 4 in [3] illogical. Furthermore, Theorems 2, 3, and 5 in [3] are specific instances of well-known results concerning the same topic for three-step iteration methods [6]. Further details on this matter are deferred to related researchers for examination.

(7). The performance of algorithms (3) and (4) is as under:

Processor: Intel Core 2 Quad with 4 GB RAM,

Digits: $100,000$,

x_0 is the initial approximation,

p is the fixed point of T,

$n(3)$ is the number of iterations for (3),

$n(4)$ is the number of iterations for (4),

$x_n(3)$ is the nth iteration of (3),

$x_n(4)$ is the nth iteration of (4),

$\varepsilon(3) = 10^{-50000}$, stopping criteria for (3),

$\varepsilon(4) = 10^{-50000}$, stopping criteria for (4),

$\delta(3) = |x_{n+1} - x_n|$ for (3),

$\delta(4) = |x_{n+1} - x_n|$ for (4),

$CPUT(3)$, CPU time for (3),

$CPUT(4)$, CPU time for (4),

$f_1 = x - (1/5) \times (1 + \cos(x)), T_1 = (1/5) \times (1 + \cos(x)), p = 0.385334547674975;$

$f_2 = \tan(x) - x, T_2 = Pi + \arctan(x), p = 4.493409e + 00;$

$f_3 = x - \cos(x), T_3 = \cos(x), p = 0.739085133215160;$

$f_4 = x + \ln(x - 2), T_4 = 2 + \exp(-x), p = 2.120028238987641;$

$f_5 = x\text{^}2 - 3, T_5 = 3/x, p = -1.732050807568877;$

$f_6 = x\text{^}3 - 3 \times x - 18, T_6 = (3 \times x + 18)\text{^}(1/3), p = 3;$

$Sr.$	f	T	x_0	$n(3)$	$n(4)$	$x_n(3)$	$x_n(4)$	$\varepsilon(3)$	$\varepsilon(4)$
1	f_1	T_1	1	16	14	$3.853345e-01$	$3.853345e-01$	$4.914806e-89013$	$5.427161e-58973$
2	f_2	T_2	4	15	14	$4.493409e+00$	$4.493409e+00$	$2.216153e-73266$	$1.899590e-80084$
3	f_3	T_3	1.7	16	16	$7.390851e-01$	$7.390851e-01$	$9.584064e-66707$	$1.290194e-88929$
4	f_4	T_4	2.1	15	14	$2.120028e+00$	$2.120028e+00$	$7.324884e-97218$	$2.325802e-78776$
5	f_5	T_5	1.6	16	16	$1.732051e+00$	$1.732051e+00$	$3.443424e-91879$	$3.443424e-91879$
6	f_6	T_6	1000	16	15	$3.000000e+00$	$3.000000e+00$	$1.905607e-54264$	$7.150590e-88827$

$\delta(3)$	$\delta(4)$	$CPUT(3)$	$CPUT(4)$
$2.302979e-44506$	$3.219263e-29485$	$548\ Sec$	$996 Sec$
$3.232845e-36633$	$6.342504e-40041$	$401 Sec$	$852 Sec$
$1.610433e-33353$	$8.771733e-44465$	$474 Sec$	$908 Sec$
$3.827502e-48609$	$1.796875e-39387$	$236 Sec$	$387 Sec$
$5.868069e-45940$	$5.868069e-45940$	$13 Sec$	$29 Sec$
$4.141311e-27132$	$2.283151e-44412$	$380 Sec$	$784 Sec$

Note 1. *(1). The three-step iteration scheme introduced in* [6] *is linearly convergent and exhibit second order of convergence under certain conditions.*

(2). The k-step version of algorithm (4):

$$\begin{array}{rcl} x_0 & \in & [a,b], \\ z_n & = & (1-\lambda)x_n + \lambda T x_n;\ \lambda = \frac{1}{1+\theta} \in (0,1], \\ y_n^{(k)} & = & T y_n^{(k-1)}, \\ x_{n+1} & = & T y_n^{(k)},\ n = 0,1,2,...,k = 0,1,2,...,y_n^{(0)} = z_n \end{array} \tag{10}$$

is linearly convergent and exhibit second order of convergence under certain conditions.

(3). All the iteration schemes (2)-(6) included in [3] *are linearly convergent and exhibit second order of convergence under certain conditions, respectively.*

(4). It is worth mentioning that the one-step iteration schemes (e.g., [2,4,7] *) are robust enough for approximating fixed points of general mappings, unless they exhibit certain convergence failures.*

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