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Article

Characterization of posinormal operators with closed ranges

Adoyo Alvince Ochieng, Benard Okelo*, Priscah Omoke

Department of Pure and Applied Mathematics, Jaramogi Oginga Odinga University of Science and Technology, Box 210-40601, Bondo-Kenya
*** Corresponding author:** Benard Okelo, bnyaare@yahoo.com

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Abstract: Characterization of posinormal operators in terms of their positivity, invertibility and numerical ranges has been done. However, characterization of these operators with regards to their closed ranges remains interesting. In this work, we characterize conditions for posinormal operators to have closed ranges. In particular, we establish an important upper norm bound criterion for posinormal operators. We show that if Q, R are normal operators in PN (H), the set of all posinormal operators acting on a Hilbert space H and suppose that the range of Q is closed with the null space of Q equal to the null space of R, then the range of R is closed. The results of this study are very useful applications in many areas, like image and signal processing. In particular, they are useful in processing signals and images used in facial recognition which are important in the identification of people in places like the airports, thus helping in enhancing security and forensic analysis.

Keywords: posinormal operator closed range; spectrum
MSC Classification: primary 47B02; secondary 47A05

1. Introduction

Operators have very important properties like norm, numerical ranges, and closed ranges that are under consideration by many authors due to their interesting and useful applications in other fields (see [1] and [2] for more details). Normal operators [3] have further subclasses like the hyponormal [4], posinormal (an operator A on a Hilbert space H is posinormal if its range is included in the range of its adjoint) [5], subnormal [6] among others. Other classes are norm-attaining [7], norm-attainable [8], convexoid [9], spectraloid [10], transaloid [11], isoloids [12] among others. These classes are related to one another in one way or another, as seen in [13]. Studies of closed range operators have also been carried out over decades, particularly the bounded linear operators on Hilbert spaces [14]. Due to many questions arising from these studies, extensions have been done in different classes of operators with consideration given in various spaces like Hilbert spaces [15]. A natural question from [16] which has not been answered states that to what extent can one characterize all composition operators on Hardy spaces that are posinormal or coposinormal? In other words, this is restated in [17] that one can give a complete characterization of composition operators and other classes of operators in a general Banach space setting? This question was further stressed by [18]. The study of the product of commuting closed range operators was initiated by [19], whereby the study characterized closed range operators in terms of their ranges, Kernels and orthogonal complements of the same. From these studies, it is clear that one property that has been considered by many authors and is still being considered is the closedness [20] of the range of these operators. In [21], the authors characterized totally (p, k)-quai-posinormal operators in terms of their numerical ranges and invertibility. In [22] the author studied posinormal operators on Hilbert spaces and came up with conditions for an operator to be in the superclass of posinormal and hyponormal. Supraposinormality of operators has also been characterized and it has been shown in [23] that the superclass contained all posinormal and coposinormal operators. The work of [24] characterized basic properties of positive normal operators in semi-Hilbertian spaces and suggested a consideration to be given to posinormal operators in terms of their closed ranges. In [25] the authors worked on powers of posinormal operators. The authors proved that posinormal operators have a closed range and it was shown that if a posinormal operator is coposinormal, then its posinormal operators with powers are also coposinormal. In [26] the authors characterized numerical ranges of posinormal operators on Hilbert spaces. It was shown that for a posinormal mapping B, $W(B)$ is nonempty and is an ellipse whose foci are eigenvalues of B. The authors discussed how the numerical range of a given operator can be obtained on Hilbert spaces.

The work of [27] characterized powers of posinormal operators on Hilbert spaces. It was shown that the powers of posinormal operators have closed ranges and it was shown that the class of posinormal operators consists of all hyponormal operators. Examples of operators with closed ranges but whose powers do not contain closed ranges were provided. The authors also studied normal and hyponormal operators where the closedness was also shown to hold. The work of [28] characterized totally posinormal operators on Hilbert spaces. Basic properties of p-posinormal operators were obtained and in particular, the authors considered the spectral continuity and range-kernel orthogonality of posinormal operators in [29]. In [30] the authors characterized composition operators with closed ranges on Dirichlet spaces. The authors in [31] gave conditions necessary for maps on Dirichlet space to have closed range. Also [32] characterized posinormal operators and products of posinormal operators with closed ranges. The authors obtained conditions for the products of two posinormal operators to be posinormal and conditions for the posinormal operators to have closed ranges. The authors in [33] also discussed the conditions necessary for the products of commuting posinormal operators with closed ranges to be posinormal with closed ranges. With regard to these questions, this study therefore seeks to carry out further characterization of closed ranges of products of commuting closed range operators [34]. In particular, we consider the class of posinormal operators and characterize upper norm bounds in terms of their closed ranges. These characterizations will help in closing some of the gaps that have been identified in this study. Finally, we give the notations as used in this study. In this work, $Ran(Q)$ is the range of an operator Q, $\mathrm{Ran}(R)$ is the range of an operator R, while $PN(H)$ is the set of all posinormal operators acting on a Hilbert space H. Moreover, $\sigma_{iso}(Q)$ is the isolated spectrum of Q.

2. Materials and methods

We provide some materials and some methods that are useful in the sequel.

Definition 1. *(The Banach's closed range theorem [35], Definition 5.7). Let Q and Z be Banach spaces. An operator $T : Q \rightarrow Z$ is said to be a closed range operator if the following conditions are equivalent*:

a) *Ran(T)* is closed in *Z*.

b) *Ran(T^*)* is closed in Q^*.

We note that *Ran(T)* is the range of *T* and *Ran*(T^*) is the range of T^* the adjoint of *T*.

Definition 2. *([36], Section 1). An operator A on a Hilbert space H is posinormal if its range is included in the range of its adjoint*.

Definition 3. *([37], Definition 2). A matrix(operator) whose range equal to the range of its adjoint or their null spaces are equal is referred to as an EP matrix(operator).*

Definition 4. *([38], Definition 2.6). An operator A which is densely defined has an inverse called Moore-Penrose Inverse (MPI) denoted by A_{MPI} which is densely defined satisfying the property that Null space of A_{MPI} is equal to the Range of orthogonal complement of A_{MPI}*

Definition 5. *([39], Definition 2.5)***.** *We define by RMM (A) = inf{$\|A\xi\|$: $\xi \in F$ (T), $\|\xi\|$ = 1.} the Reduced Minimum Modulus (RMM) where F (T) is equal to the intersection of the null space of the orthogonal complement of A and the domain of A.*

3. Results

We start by determining conditions for posinormal operators to have closed ranges. We begin with the following proposition.

Proposition 1. *Let Q $\in$ PN (H) be such that it is densely defined on H then Ran(Q) is closed.*

Proof of Proposition 1. From the definition of a densely defined operator, it is known from [40] that the closure of the domain of *Q* is equal to *H*. Moreover, *Q* has an adjoint Q^* which is unique by closed graph theorem (CGT) so it follows that a bounded operator is a cloed operator. Since *Q* is closed then there exists a sequence ξn in H such that $Q\xi_n \to \xi$, for all $\xi \in H$. Hence, *Ran(Q)* is closed. □

Lemma 1. *Let Q $\in$ PN (H) be having a bounded Q_{MPI}. Then Ran(Q) is closed.*

Proof of Lemma 1. Let *Q* be having a bounded Q_{MPI}. Then it is known from [41] that the null space of *Q* is equal to the range of the orthogonal complement of Q. So the orthogonal projection (OP) $PRan(Q_{MPI})(\xi)$ for all ξ in the domain of *Q* is equal to $Q_{MPI}Q$ acting on ξ. But $Ran(Q_{MPI})$ is closed [42]. For boundedness [43], let ξ be in the domain of Q_{MPI}. Then we have some ξ_0 in the domain of Q_{MPI} such that $\|Q_{MPI}\xi_0\| \leq k \|\xi_0\|$, for some ξ_0 in the domain of Q_{MPI}. Hence, Q_{MPI} is bounded [44]. Finally, since Q_{MPI} is bounded and closed then from the Inverse mapping Theorem [45], *Q* is also closed. Hence, from Proposition 1 and an assertion in [46] and [47] *Ran(Q)* is closed. □

Theorem 1. *Let Q ∈ PN (H) be densely defined on H. If RMM (Q) is equal to the reciprocal of Q_{MPI} and RMM (Q) = RMM (Q^*Q) = RMM (Q^2) then Ran(Q) is closed.*

Proof of Theorem 1. From Proposition 1, Lemma 1 and a characterization in [48] we have that *Q* is closed and bounded. Moreover, Q_{MPI} is also closed from a proof in [49]. Indeed, we $\sup\{\|Q_{MPI}\xi\| : \|\xi\| = 1\} = RMM\,(Q)^{-1}$, for all ξ in the domain of Q_{MPI}. For the second part of the proof let *RMM (Q)* be nonzero. From Lemma 1 we have that *Ran(Q)* is closed. But from [50] if *RMM (Q)* is nonzero then *Ran(Q)* is not closed and *RMM (Q^*Q)* is equal to zero. Now consider *RMM (Q)* to be strictly greater that zero.

□

Remark 1. *Every positive $Q \in PN(H)$ with a bounded spectrum has a closed range [51].*

Corollary 1. *Let $Q \in PN(H)$ be self-adjoint then $Ran(Q^*)$ is closed.*

Proof of Corollary 1. The proof follows from Theorem 1 and the fact that Q is self-adjoint. □

Corollary 2. *Let $Q \in PN(H)$ be such that $RMM(Q) = RMM(Q^*)$ then $Ran(Q)$ is closed.*

Proof of Corollary 2. From Remark 1 we know that Q is self-adjoint and hence Q_0. Therefore, $RMM(Q) = RMM(Q^*)$ since from Lemma 1 it follows that $RMM(QQ^*) = RMM(Q^*Q)$. Since $Ran(Q^*)$ is closed then $Ran(Q)$ is closed because $Q = Q^*$. The rest follows from Corollary 2 and the technique of [52]. □

Next, we characterize when the posinormal operators are strictly posinormal as seen in the next Proposition.

Proposition 2. *Let $Q \in PN(H)$ be self-adjoint. If $\alpha \in \sigma_{iso}(Q)$ is an eigenvalue then $Ran(Q)$ is closed.*

Proof of Proposition 2. Let $\alpha \in \sigma_{iso}(Q)$. It follows from [53] that zero is in σ_{iso}(Q). Of course, from Remark 1 we have that σ(Q) is nonempty hence $\sigma_{iso}(Q)$ is also nonempty. From Theorem 1 $RMM(Q) > 0$ and Corollary 3.5 asserts that $Q = Q^*$. The fact that α is an eigenvalue follows from the assertion that zero is in $\sigma_{iso}(Q - \alpha I)^2$. Since $\sigma_{iso}(Q - \alpha I)^2$ is positive so Q has a bounded spectrum of isolated points. From Remark 1 every $Q \in PN(H)$ with a bounded spectrum has a closed range. Hence, $Ran(Q)$ is closed. □

Lemma 2. *Let $Q \in PN(H)$ be normal. Then $Ran(Q)$ is closed.*

Proof of Lemma 2. Since Q is normal then from Corollary 1, we have that Q is self-adjoint. Also, we have $RMM(Q) = RMM(Q^*)$. So, $Ran(Q)$ is closed if and only if it is normal and self-adjoint. Next, we give our main result which is a new characterization of the closed ranges of posinormal operators in-terms of upper norm bounds. □

Theorem 2. *Consider $Q, R \in PN(H)$ be normal. Suppose that $Ran(Q)$ is closed with null space of Q equal to the null space of R then $Ran(R)$ is closed if $\|Q\xi\| \leq M\|R\xi\|$, for all $\xi \in H$.*

Proof of Theorem 2. From the Closed Graph Theorem (CGT), the null space of any operator is closed if and only if we have a constant $m > 0$ such that $Q\xi = Q\xi_0$ for all $\xi \in H$ and some $\xi_0 \in H$ and $\|\xi 0\| \leq m\|Q\|$. Now R, Q are one to one from Remark 1 onto the orthogonal complement of the null space of Q. So, the null space of R is a subset of the null space of Q. Let $0 \in \sigma_{iso}(Q)$ as in Proposition 2 then $\sigma_{iso}(Q)$ is nonempty and a subset of the orthogonal complement of the null space of Q. Therefore, R acting on the orthogonal complement of the null space of Q is closed in H. Hence, $Ran(R)$ is closed. □

Example 1. *Consider $Q, R \in B(H)$ be normal. Suppose that $Ran(Q)$ is closed with null space of Q equal to the null space of R then $Ran(R)$ is closed if $\|Q\xi\| \geq M\|R\xi\|$, for all $\xi \in H$.*

Remark 2. *Consider $Q, R \in B(H)$. Suppose that R, Q are norm equivalent then the null space of R is equal to the null space of Q. It follows that Ran(R) is closed if and only if Ran(Q) is closed.*

4. Applications

Operator theory has very useful applications as seen in [54] many areas like image and signal processing. In particular, applications of posinormal operators are immense in different fields. These operators are useful in quantum mechanics whereby they are significant in the estimation of the distances moved by the electrons in the orbitals [55]. This enables quantum physicists in determining the ground state energies of these electrons which are in turn useful in processing signals and images used in facial recognition which are important in the identification [56] of people in places like the airports, thus helping in enhancing security and forensic analysis.

5. Conclusion

In many studies, characterization of posinormal operators in terms of their positivity, invertibility and numerical ranges has been done. However, characterization of these operators with regards to their closed ranges remains interesting. In this work, we have characterized conditions for posinormal operators to have closed ranges. In particular, we have established an important upper norm bound criterion for posinormal operators when they have closed ranges. We have shown that if $Q, R \in PN(H)$ are normal and suppose that $Ran(Q)$ is closed with null space of Q equal to the null space of R, then $Ran(R)$ is closed if $\|Q\xi\| \leq M\|R\xi\|$, for all $\xi \in H$. The results of this study are useful in application in quantum mechanics and portfolio optimization in financial mathematics. We recommend a further study to be carried out on the characterization of the products of posinormal operators when they are unbounded in a general Banach space setting. This work is useful in image and signal processing which are important in facial recognition and forensic analysis.

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