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Article

Regularization of the Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain

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Abstract: In this paper, a regularized solution to the Cauchy problem for matrix factorization of the Helmholtz equation in a three-dimensional unbounded domain is constructed explicitly based on the Carleman matrix. When solving applied problems, in addition to an approximate solution, the derivative of the approximate solution is found. It is assumed that the solution to the problem exists and is continuously differentiable in a closed domain with precisely specified Cauchy data. An explicit formula for continuing the solution and its derivative is established, as well as a regularization formula for the case when, under the specified conditions, instead of the original Cauchy data, their continuous approximations with a specified error in the uniform metric are given. As a result, the stability of the solution to the Cauchy problem in the classical sense is estimated.

Keywords: ill-posed tasks; the Cauchy problem; conditional correctness; explicit formula; unbounded domain
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1. Introduction

At present, the theory of correctly and ill-posed problems, most of which have practical significance, is rapidly developing. Obviously, the theory of ill-posed problems is an apparatus of scientific research for many scientific directions and studies. The concept of a correctly posed problem was first introduced by Hadamard [1], and he asserted that any mathematical problem corresponding to some physical or technological problem must be correctly posed [1]. When it comes to ill-posed problems, the following question arises: What do we mean by an approximate solution? It must be defined so as to be stable with small changes in the initial information. The second question: What algorithms are correct for constructing such solutions? The answer to this question can be found in the work [2].

The term conditional correctness first appeared in Tikhonov's scientific research [2], then in works [3,4]. When an ill-posed problem is correct according to Tikhonov, the existence of a solution and its belonging to the correctness set are assumed in the problem statement itself. After the uniqueness and stability theorems are established in the study of the conditional correctness of ill-posed problems, the question of constructing optimal solution methods arises. We are well aware that the Cauchy problem for any elliptic equations and for systems of elliptic equations is considered ill-posed (see, for example, [1–9]). Boundary value problems for various equations were considered in [10–13].

Using the methodology of works [3–4,8–9], in this work we will construct the Carleman matrix and a regularized solution based on it. In this paper, we find a regularized solution to the Cauchy problem in explicit form for matrix factorizations of the Helmholtz equation of an unbounded domain. Our approximate solution formula also includes the construction of a family of fundamental solutions of the Helmholtz operator in space. This family is represented by some entire function depending on the dimension of the space. In this study, based on works [14–16], we obtained better results due to the $K(z)$, function. Based on these results, we were able to obtain effective results in finding an approximate solution based on the Carleman matrix. The Carleman matrix or the Carleman function are also constructed in works. This helped to get good results when finding an approximate solution based on the Carleman matrix. The Carleman matrix or Carleman function for some elliptic equations and systems was considered in the following studies [17–20].

In many correct problems for elliptic equations, it is not possible to calculate the values of the vector function on the entire boundary. Because of this, the problem of restoring the solution of elliptic-type systems is one of the topical problems. At present, there is a special interest in problems of this type and their applications.

Let $\mathbb{R}^3$ be the three-dimensional real Euclidean space.

$$\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3, \ \ \eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3,$$

$$\zeta' = (\zeta_1, \zeta_2) \in \mathbb{R}^2, \ \ \eta' = (\eta_1, \eta_2) \in \mathbb{R}^2.$$

$\Omega \subset \mathbb{R}^3$ is an unbounded simply-connected domain with piecewise smooth boundary consisting of the plane $D\colon \eta_3 = 0$ and a smooth surface Σ lying in the half-space $\eta_3 > 0$, i.e., $\partial\Omega = \Sigma \cup D$.

Next, we will use the following notations:

$$r = |\eta - \zeta|, \qquad \alpha = |\eta' - \zeta'|, \qquad z = i\sqrt{a^2 + \alpha^2} + \eta_3, a \geq 0,$$

$$\partial_\zeta = (\partial_{\zeta_1}, \partial_{\zeta_2}, \partial_{\zeta_3})^T, \ \partial_\zeta \to \chi^T, \ \chi^T = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix}, \text{transposed vector } \chi,$$

$$W(\zeta) = (W_1(\zeta), \dots, W_n(\zeta))^T, \ \ v^0 = (1, \dots, 1) \in \mathbb{R}^n, \ \ n = 2^m, \ \ m = 3,$$

$$E(u)=\begin{Vmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & u_n \end{Vmatrix} \text{−diagonal matrix, } u=(u_1,\dots,u_n)\in\mathbb{R}^n.$$

We consider a bounded simply-connected domain $\Omega\subset\mathbb{R}^m$, having a piecewise smooth boundary $\partial\Omega=\Sigma\cup D$, where Σ is a smooth surface lying in the half-space Σ and D is the plane $\eta_m=0$.

Let $P(\chi^T)$ be a square matrix of dimension $(n\times n)$ for which the following holds:

$$P^*(\chi^T)P(\chi^T)=E((|\chi|^2+\lambda^2)v^0),$$

where $P^*(\chi^T)$ means the Hermitian conjugate matrix of $P(\chi^T)$, $\lambda\in\mathbb{R}$, the elements of the matrix $P(\chi^T)$ consist of a set of linear functions with constant coefficients from the complex plane $\mathbb{C}$.

Let's consider the following system of equations:

$$P(\partial_\zeta)\,W(\zeta)=0, \tag{1}$$

in the domain Ω, where $P(\partial_\zeta)$ is the matrix differential operator of the first-order.

Let's assume a set:

$$S(\Omega)=\{W:\overline{\Omega}\to\mathbb{R}^n\},$$

here W is considered continuous on $\overline{\Omega}=\Omega\cup\partial\Omega$ and W is a solution of system (1).

2. Statement of the Cauchy problem

Suppose $f:\Sigma\to\mathbb{R}^n$ be a continuous given function on Σ.

Let $W(\eta)\in S(\Omega)$ and

$$W(\eta)|_\Sigma=f(\eta),\quad \eta\in\Sigma. \tag{2}$$

Our main goal is to determine the function $W(\eta)$ in the domain Ω, based on its known values on Σ.

If $W(\eta)\in S(\Omega)$, then the following Cauchy type integral formula:

$$W(\zeta)=\int_{\partial\Omega}L(\eta,\zeta;\lambda)W(\eta)ds_\eta,\quad \zeta\in\Omega, \tag{3}$$

is valid and

$$L(\eta,\zeta;\lambda)=(E(\Gamma_3(\lambda r)v^0)P\times(\partial_\zeta))P(t^T),$$

where $t=(t_1,t_2,t_3)$ shows the unit exterior normal, which is drawn at a point η on the surface $\partial\Omega$ and $\Gamma_3(\lambda r)$ −is the fundamental solution of the Helmholtz equation (see [16]), which has the following form:

$$\Gamma_3(\lambda r)=-\frac{e^{i\lambda r}}{4\pi r}. \tag{4}$$

Let $K(z)$ be an entire function taking real values z, $(z=a+ib,\ a,b$ −real numbers) for which the following is true [8,9]:

$$K(a) \neq 0, \quad \sup_{b \geq 1} \left| b^p K^{(p)}(z) \right| = B(a,p) < \infty, \\ -\infty < a < \infty, \quad p = 0,1,2,3. \tag{5}$$

The function $\Psi(\eta, \zeta; \lambda)$ for $\eta \neq \zeta$ is defined as follows:

$$\Psi(\eta, \zeta; \lambda) = -\frac{1}{2\pi^2 K(\zeta_3)} \int_0^{\infty} \mathrm{Im} \left[\frac{K(z)}{z - \zeta_3} \right] \frac{\cos(\lambda a)}{\sqrt{a^2 + \alpha^2}} da, \tag{6}$$

Equation (3) remains unchanged; we represent the function $\Gamma_3(\lambda r)$ as follows:

$$\Psi(\eta, \zeta; \lambda) = \Gamma_3(\lambda r) + G(\eta, \zeta; \lambda), \tag{7}$$

where $G(\eta, \zeta; \lambda)$ is a regular solution with respect to the variable η, including the point $\eta = \zeta$.

In this case (3) is depicted as follows:

$$W(\zeta) = \int_{\partial\Omega} L(\eta, \zeta; \lambda) W(\eta) ds_\eta, \quad \zeta \in \Omega, \tag{8}$$

where:

$$L(\eta, \zeta; \lambda) = (E(\Psi(\eta, \zeta; \lambda) v^0) P^*(\partial_\zeta)) P(t^T).$$

We generalize Equation (8) to an unbounded domain Ω.

Suppose $\Omega \subset \mathbb{R}^3$ be an unbounded domain, and its boundary $\partial\Omega$ be piecewise smooth (extending to infinity).

And also let

$$\Omega_R = \{\eta : \eta \in \Omega, |\eta| < R\}, \ \ \Omega_R^\infty = \Omega \backslash \Omega_R, \ R > 0.$$

Theorem 1. *Suppose that $W(\eta) \in S(\Omega)$, and Ω is a finitely connected unbounded domain in three-dimensional space with piecewise smooth boundary $\partial\Omega$.*
If for a fixed $\in \Omega$ the following is true

$$\lim_{R \to \infty} \int_{\Omega_R^\infty} L(\eta, \zeta; \lambda) W(\eta) ds_\eta = 0, \tag{9}$$

then the integral representation Equation (8) will be true.

Proof. It is known that for a fixed $\zeta \in \Omega(|\zeta| < R)$, relying on the integral representation Equation (8) we have:

$$\int_{\partial\Omega} L(\eta, \zeta; \lambda) W(\eta) ds_\eta = \int_{\partial\Omega_R} L(\eta, \zeta; \lambda) W(\eta) ds_y + \\ + \int_{\partial\Omega_R^\infty} L(\eta, \zeta; \lambda) W(\eta) ds_\eta = W(\zeta) + \int_{\partial\Omega_R^\infty} L(\eta, \zeta; \lambda) W(\eta) ds_\eta, \zeta \in \Omega_R.$$

Due to the limit condition Equation (9), as $R \to \infty$, we obtain the integral representation Equation (8).

Suppose that the unbounded domain Ω is defined as follows:

$$0 < \eta_3 < h, h = \frac{\pi}{\rho}, \rho > 0,$$

and most importantly $\partial\Omega$ extends to infinity.

Let for any $d_0 > 0$ for area $\partial\Omega$ the following growth condition be true:

$$\int_{\partial\Omega} \exp[-d_0\rho_0|\eta'|]ds_\eta < \infty, 0 < \rho_0 < \rho. \tag{10}$$

Suppose $W(\eta) \in S(\Omega)$ that it satisfies the boundary growth condition:

$$|W(\eta)| \leq \exp[\exp\rho_2|\eta'|],\ \ \rho_2 < \rho, \eta \in \Omega. \tag{11}$$

In (6) we put:

$$K(z) = \exp\left[-di\rho_1\left(z-\frac{h}{2}\right) - d_1 i\rho_0\left(z-\frac{h}{2}\right)\right],$$

$$K(\zeta_3) = \exp\left[d\cos\rho_1\left(\zeta_3-\frac{h}{2}\right) + d_1\cos i\rho_0\left(\zeta_3-\frac{h}{2}\right)\right], \tag{12}$$

$$0 < \rho_1 < \rho,\ \ 0 < \zeta_3 < h,$$

where

$$d = 2c\exp(\rho_1|\zeta'|), d_1 > \frac{d_0}{\cos\left(\rho_0\frac{h}{2}\right)},\ \ c \geq 0, d > 0.$$

Then the integral representation Equation (8) is true.

For a fixed $\zeta \in \Omega$ and $\eta \to \infty$, we estimate the functions $\Psi(\eta,\zeta;\lambda)$, $\frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_j}$, $j = \overline{1,2}$ and $\frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_3}$.

To estimate $\frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_j}$, we use the equality:

$$\frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_j} = \frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial s}\frac{\partial s}{\partial\eta_j} = 2(\eta_j-\zeta_j)\frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial s},\ \ j = \overline{1,2}. \tag{13}$$

And so,

$$\left|\exp\left[-di\rho_1\left(z-\frac{h}{2}\right) - d_1 i\rho_0\left(z-\frac{h}{2}\right)\right]\right| =$$
$$= \exp\mathrm{R}e\left[-di\rho_1\left(z-\frac{h}{2}\right) - d_1 i\rho_0\left(z-\frac{h}{2}\right)\right] =$$
$$= \exp\left[-d\rho_1\sqrt{a^2+\alpha^2}\cos\rho_1\left(\eta_3-\frac{h}{2}\right) - \right.$$
$$\left. d_1\rho_0\sqrt{a^2+\alpha^2}\cos\rho_0\left(\eta_3-\frac{h}{2}\right)\right].$$

As:

$$-\frac{\pi}{2} \leq -\frac{\rho_1}{\rho}\cdot\frac{\pi}{2} \leq \frac{\rho_1}{\rho}\frac{\pi}{2} < \frac{\pi}{2},$$
$$-\frac{\pi}{2} \leq -\frac{\rho_1}{\rho}\frac{\pi}{2} \leq \rho_0\left(y_3-\frac{h}{2}\right) \leq \frac{\rho_1}{\rho}\frac{\pi}{2} < \frac{\pi}{2}.$$

Consequently,

$$\cos\rho\left(\eta_3-\frac{h}{2}\right) > 0, \cos\rho_0\left(\eta_3-\frac{h}{2}\right) \geq \cos\frac{h\rho_0}{2} > \delta_0 > 0.$$

It does not vanish in the region Ω and:

$$|\Psi(\eta,\zeta;\lambda)| = O[\exp(-\varepsilon\rho_1|\eta'|)],\ \ \varepsilon > 0,\ \ \eta \to \infty,\ \ \ \eta \in \Omega \cup \partial\Omega,$$
$$\left|\frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_j}\right| = O[\exp(-\varepsilon\rho_1|\eta'|)],\ \ \varepsilon > 0,\ \ \eta \to \infty,\ \ \eta \in \Omega \cup \partial\Omega,\ \ j = 1,2.$$
$$\left|\frac{\partial\Psi(\eta,\zeta;\lambda)}{\partial\eta_3}\right| = O[\exp(-\varepsilon\rho_1|\eta'|)],\ \ \varepsilon > 0,\ \ \eta \to \infty,\ \ \eta \in \Omega \cup \partial\Omega.$$

We now choose ρ_1 with the condition $\rho_2 < \rho_1 < \rho$. In this case, condition Equation (10) will be valid, and the integral representation Equation (8) will take place. □

Next, for convenience, we weaken condition Equation (12).

Let us denote by $S_\rho(\Omega)$ the following growth condition:

$$S_\rho(\Omega) = \{W(\eta)\colon W(\eta) \in S(\Omega), |W(\eta)| \le \exp[o[\exp\rho|\eta_1|]],\ \eta \to \infty,\ \eta \in \Omega\}. \quad (14)$$

The following is valid:

Theorem 2. *Suppose* $W(\eta) \in S_\rho(\Omega)$ *and the following is true*:

$$|W(\eta)| \le C\exp\left[c\cos\rho_1\left(\eta_3 - \frac{h}{2}\right)\exp(\rho_1|\eta'|)\right], \quad c \ge 0, 0 < \rho_1 < \rho, \eta \in \partial\Omega, \quad (15)$$

where C *—is a constant. Then the integral representation Equation (8) is valid.*

Proof. Next, we divide the region Ω by the line $\eta_3 = \frac{h}{2}$ into two corresponding the following regions:

$$\Omega_1 = \left\{\eta\colon 0 < \eta_3 < \frac{h}{2}\right\} \text{ and } \Omega_2 = \left\{\eta\colon \frac{h}{2} < \eta_3 < h\right\}.$$

First, let us consider the domain Ω_1. To do this, we substitute the functions $K_1(z)$ into equality Equation (6) instead of the functions $K(z)$

$$K_1(z) = K(z)\exp\left[-\delta\, i\tau\left(z - \frac{h}{2}\right) - \delta_1 i\rho\left(z - \frac{h}{2}\right)\right], \quad \rho < \tau < 2\rho, \delta > 0, \delta_1 > o, \quad (16)$$

Here the function $K(z)$ is directly determined from Equation (12). In these cases, condition Equation (10) will be true.

And so,

$$\left|\exp\left[-i\tau\left(z - \frac{h}{4}\right) - \delta_1 i\rho\left(z - \frac{h}{4}\right)\right]\right| = $$
$$= \exp\left[-\delta\tau\sqrt{a^2 + \alpha^2}\cos\tau\left(\eta_3 - \frac{h}{4}\right)\right] =$$
$$= \exp\left[-\delta\tau\sqrt{a^2 + \alpha^2}\right] \le \exp[-\delta\exp\tau|\eta'|],$$

as

$-\frac{\pi}{2} \le -\tau\frac{\pi}{4} \le \tau\left(\eta_3 - \frac{h}{4}\right) \le \tau\frac{\pi}{2} < \frac{h}{2}$ and $\cos\tau\left(\eta_3 - \frac{h}{4}\right) \ge \cos\tau\frac{h}{4} \ge \delta_0 > 0$.

We denote the corresponding $\Psi(\eta,\zeta;\lambda)$ by $\Psi^+(\eta,\zeta;\lambda)$.

As

$$\cos\tau\left(\eta_3 - \frac{h}{4}\right) \ge \delta_0, \quad \eta \in \Omega_1 \cup \partial\Omega_1,$$

then for fixed $\zeta \in \Omega_1, \eta \in \Omega_1 \cup \partial\Omega_1$, for $\Psi^+(\eta,\zeta;\lambda)$ and its derivatives are true asymptotic estimates.

$$|\Psi^{+}(\eta,\zeta;\lambda)| = O[\exp(-\delta_0 \exp(\tau|\eta'|)],\ \ \eta \to \infty,\ \ \rho < \tau < 2\rho,$$

$$\left|\frac{\partial \Psi^{+}(\eta,\zeta;\lambda)}{\partial \eta_j}\right| = O[\exp(-\delta_0 \exp(\tau|\eta'|)],\ \ \eta \to \infty,\ \ \rho < \tau < 2\rho,\ \ j = 1,2.$$

$$\left|\frac{\partial \Psi^{+}(\eta,\zeta;\lambda)}{\partial \eta_3}\right| = O[\exp(-\delta_0 \exp(\tau|\eta'|)],\ \ \eta \to \infty,\ \ \rho < \tau < 2\rho.$$

Suppose that $W(\eta) \in S_\rho(\Omega_1)$, and in the domain Ω_1 the following is true

$$|W(\eta)| \le C\exp[\exp(2\rho - \varepsilon)|\eta'|],\ \ \varepsilon > 0. \tag{17}$$

We choose τ the inequality $2\rho - \varepsilon < \tau < 2\rho$ in Equation (16).

It is obvious that for the region Ω_1 the condition Equation (16) will be satisfied, then therefore the following integral representation is valid

$$W(\zeta) = \int_{\partial\Omega_1} L(\eta,\zeta;\lambda)W(\eta)ds_\eta,\ \ \ \zeta \in \Omega_1. \tag{18}$$

where

$$L(\eta,\zeta;\lambda) = (E(\Psi^{+}(\eta,\zeta;\lambda)v^0)P^*(\partial_\zeta))P(t^T).$$

If $W(\eta) \in S_\rho(\Omega_2)$ in Ω_2 satisfies (15), then for $2\rho - \varepsilon < \tau < 2\rho$ we obtain the following

$$W(\zeta) = \int_{\partial\Omega_2} L(\eta,\zeta;\lambda)W(\eta)ds_\eta,\ \ \ \zeta \in \Omega_2. \tag{19}$$

where

$$L(\eta,\zeta;\lambda) = (E(\Psi^{-}(\eta,\zeta;\lambda)v^0)P^*(\partial_\zeta))P(t^T).$$

Here the function $\Psi^{-}(\eta,\zeta;\lambda)$ is represented by formula (6), in which the function $K(z)$ is taken as a function of $K_2(z)$:

$$K_2(z) = K(z)\exp\left[-\delta i\tau(z - h_1) - \delta_1 i\rho\left(z - \frac{h}{2}\right)\right], \tag{20}$$

where

$$h_1 = \frac{h}{2} + \frac{h}{4},\ \ \frac{h}{2} < \eta_3 < h,\ \ \frac{h}{2} < \zeta_3 < h_1,\ \ \delta > 0,\ \ \delta_1 > 0.$$

In these formulas, the integrals (according to Equation (11)) converge uniformly for $\delta \ge 0$, when $W(\eta) \in S_\rho(\Omega)$. In these formulas we put $\delta = 0$ and, combining the formulas obtained, we find

$$W(\zeta) = \int_{\partial\Omega} L(\eta,\zeta;\lambda)W(\eta)ds_\eta,\ \ \ \zeta \in \Omega,\ \ \ \zeta_3 \ne \frac{h}{2}, \tag{21}$$

where

$$L(\eta,\zeta;\lambda) = \left(E\left(\widetilde{\Psi}(\eta,\zeta;\lambda)v^0\right)P^*\left(\partial_\zeta\right)\right)P(t^T).$$

(Note that here the integrals over the cross section $\eta_3 = \frac{h}{2}$ cancel each other out)

Here the function $\widetilde{\Psi}(\eta,\zeta;\lambda)$ will be determined on the basis of Equation (6), and the function $K(z)$ is determined from Equation (16), where $\delta = 0$. Based on the continuation principle, Equation (21) will be true for$\forall\zeta \in \Omega$. Taking into account condition Equation (18), the integral representation Equation (21) will also be true for $\forall\delta_1 \geq 0$. Assuming $\delta_1 = 0$, we obtain the complete proof of the theorem. □

In the integral representation Equation (6), choosing functions $K(z)$ and $K(\zeta_3)$ as follows:

$$\begin{gathered} K(z) = \frac{1}{(z-\zeta_3+2h)^2}\exp(\sigma z), \\ K(\zeta_3) = \frac{1}{(2h)^2}\exp(\sigma\zeta_3), \quad 0 < \zeta_3 < h, \quad h = \frac{\pi}{\rho}, \end{gathered} \tag{22}$$

we get

$$\Psi_\sigma(\eta,\zeta;\lambda) = -\frac{e^{-\sigma\zeta_3}}{\pi^2(2h^2)^{-1}}\int_0^\infty \mathrm{Im}\frac{\exp(\sigma z)}{(z-\zeta_3+2h)^2(z-\zeta_3)}\frac{\cos(\lambda a)}{\sqrt{a^2+\alpha^2}}da. \tag{23}$$

Then the integral formula Equation (8) has the following form:

$$W(\zeta) = \int_{\partial\Omega} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta, \;\; \zeta \in \Omega, \tag{24}$$

where

$$L_\sigma(\eta,\zeta;\lambda) = E(\Psi_\sigma(\eta,\zeta;\lambda)v^0)P^*(\partial_\zeta)(P(t^T).$$

3. Approximate solution of the Cauchy problem

Theorem 3. *Assume that $W(\eta) \in S_\rho(\Omega)$ and the following is true*

$$|W(\eta)| \leq M, \;\; \eta \in D. \tag{25}$$

If

$$W_\sigma(\zeta) = \int_\Sigma L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta, \;\; \eta \in \Omega, \tag{26}$$

then the following estimates are true:

$$|W(\zeta) - W_\sigma(\zeta)| \leq K_\rho(\lambda,\zeta)\sigma M e^{-\sigma\zeta_3}, \;\; \zeta \in \Omega, \tag{27}$$

$$\left|\frac{\partial W(\zeta)}{\partial\zeta_j} - \frac{\partial W_\sigma(\zeta)}{\partial\zeta_j}\right| \leq K_\rho(\lambda,\zeta)\sigma M e^{-\sigma\zeta_3}, \;\; \sigma > 1, \;\; \zeta \in \Omega, \;\; j = 1,2,3. \tag{28}$$

Where $K_\rho(\lambda,\zeta)$ represents the bounded functions on compact subsets of the domain Ω.

Proof. To do this, we first estimate Equation (27). Based on the integral representation Equation (24), as well as the equality Equation (26), we obtain the following

$$\begin{gathered} W(\zeta) = \int_\Sigma L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta + \int_D L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta = \\ = W_\sigma(\zeta) + \int_D L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta, \quad \zeta \in \Omega. \end{gathered}$$

Based on Equation (25), we next estimate the following

$$|W(\zeta)-W_\sigma(\zeta)| \le \left|\iint_D L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta\right| \le \\ \le \int_D |L_\sigma(\eta,\zeta;\lambda)||W(\eta)|ds_\eta \le M\int_D |L_\sigma(\eta,\zeta;\lambda)|ds_\eta,\ \ \zeta\in\Omega. \tag{29}$$

Next, we estimate the integrals $\int_D |\Psi_\sigma(\eta,\zeta;\lambda)|ds_\eta$, $\int_D \left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_j}\right| ds_\eta, j=\overline{1,2}$ and $\int_D \left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\zeta_3}\right| ds_\eta$ on the part D of the plane $\eta_3=0$.

Now separating the imaginary part of equality (23), we finally obtain the following

$$\Psi_\sigma(\eta,\zeta;\lambda)=\frac{e^{\sigma(\eta_3-\zeta_3)}}{\pi^2(2h^2)^{-1}}\left[\int_0^\infty\left(\frac{(-\alpha_1^2+\beta_1^2+2\beta_1\beta)\cos\sigma\alpha_1}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}+\right.\right. \\ \left.\left.+\frac{(2\alpha_1^2\beta_1+\alpha_1^2\beta-\beta_1^2\beta)}{(\alpha_1^2+\beta_1^2)^2(\alpha_1^2+\beta^2)}\frac{\sin\sigma\alpha_1}{\alpha_1}\right)\cos(\lambda a)da\right], \tag{30}$$

where

$$\alpha_1^2=a^2+\alpha^2,\ \ \beta=\eta_3-\zeta_3,\ \ \beta_1=\eta_3-\zeta_3+2h.$$

Then, based on Equation (30), we have

$$\int_D |\Psi_\sigma(\eta,\zeta;\lambda)|ds_\eta \le K_\rho(\lambda,\zeta)\sigma e^{-\sigma\zeta_3},\ \ \sigma>1,\ \ \zeta\in\Omega. \tag{31}$$

Next, we will use the following equality

$$\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_j}=\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial s}\frac{\partial s}{\partial\eta_j}=2(\eta_j-\zeta_j)\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial s}, \\ s=\alpha^2,\ \ j=\overline{1,2}. \tag{32}$$

Based on Equation (30) and equality Equation (32), we obtain the following

$$\int_D\left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_j}\right|ds_\eta \le K_\rho(\lambda,\zeta)\sigma e^{-\sigma\zeta_3},\ \ \sigma>1,\ \ \zeta\in\Omega,\ \ j=1,2. \tag{33}$$

Now, we will estimate $\int_D \left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_3}\right| ds_\eta$.

And here too, based on Equation (30), we obtain the following

$$\int_D\left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_3}\right|ds_\eta \le K_\rho(\lambda,\zeta)\sigma e^{-\sigma\zeta_3},\ \ \sigma>1,\ \ \zeta\in\Omega, \tag{34}$$

From inequalities Equation (31), Equations (33) and (34), taking into account Equation (29), we finally obtain an estimate Equation (27).

Now it remains to prove the inequality (28). To do this, we will take the derivatives of equalities Equations (24) and (26) with respect to ζ_j, $(j=\overline{1,3})$, and as a result we will estimate the following:

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j}-\frac{\partial_\sigma W(\zeta)}{\partial \zeta_j}\right| \le \left|\int_D \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| \le$$
$$\le \int_D \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| |W(\eta)| ds_\eta \le M \int_D \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta, \tag{35}$$
$$\zeta \in \Omega,\ \ j=\overline{1,3}.$$

In order to prove Equation(35), we will evaluate here $\int_D \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta,\ j=\overline{1,2}$ and $\int_D \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_3}\right| ds_\eta$, on the part D of the plane $\eta_3=0$.
In order to evaluate the first integrals, we will use the equality

$$\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_1}=\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial s}\frac{\partial s}{\partial \zeta_j}=-2(\eta_j-\zeta_j)\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial s},$$
$$s=\alpha^2,\ \ j=\overline{1,2}. \tag{36}$$

Based on equalities Equations (30) and (36), we finally obtain

$$\int_D \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta \le K_\rho(\lambda,\zeta)\sigma e^{-\sigma\zeta_3},\ \ \sigma>1,\ \ \zeta\in\Omega,\ \ j=\overline{1,2}. \tag{37}$$

And now it remains to evaluate the integral $\int_D \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_3}\right| ds_\eta$.
Based on Equation (30), we obtain as a result

$$\int_D \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_3}\right| ds_\eta \le K_\rho(\lambda,\zeta)\sigma e^{-\sigma\zeta_3},\ \ \sigma>1,\ \ \zeta\in\Omega. \tag{38}$$

From Equations (37) and (38), based on Equation (35), we finally obtain the validity of estimate Equation (28). □
Corollary 1. *For each $\zeta \in \Omega$ the following limit equalities hold:*

$$\lim_{\sigma\to\infty} W_\sigma(\zeta)=W(\zeta),\ \ \lim_{\sigma\to\infty}\frac{\partial W_\sigma(\zeta)}{\partial \zeta_j}=\frac{\partial W(\zeta)}{\partial \zeta_j},\ \ j=1,2,3.$$

We define $\overline{\Omega}_\varepsilon$ as

$$\overline{\Omega}_\varepsilon=\{(\zeta_1,\zeta_2,\zeta_3)\in\Omega,\ \ a>\zeta_3\ge\varepsilon,\ q=\max_D\psi(\zeta'),\ \ 0<\varepsilon<q\}.$$

Here $\psi(\zeta')$ −is a surface. It is easy to see that the set $\overline{\Omega}_\varepsilon \subset \Omega$ is compact.
Corollary 2. *If $\zeta \in \overline{\Omega}_\varepsilon$, then the families of vector functions $\{W_\sigma(\zeta)\}$ and $\left\{\frac{\partial W_\sigma(\zeta)}{\partial \zeta_j}\right\}$ are satisfied uniformly as σ→∞, i.e.:*

$$W_\sigma(\zeta) \rightrightarrows W(\zeta),\ \ \frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} \rightrightarrows \frac{\partial W(\zeta)}{\partial \zeta_j},\ \ j=\overline{1,3}.$$

We should note separately that the set $E_\varepsilon=\Omega\backslash\overline{\Omega}_\varepsilon$ is a boundary layer for this problem, as in the theory of singular perturbations, where there is no uniform convergence.

Let us assume that the boundary of the domain Ω belongs to the hyperplane $\eta_3=0$ and a smooth surface S, which extends to infinity and lies in the following layer:

$$0 < \eta_3 < h, \ \ h = \frac{\pi}{\rho}, \ \ \rho > 0.$$

Suppose that the surface Σ (or the curve at $m = 2$) is given by the equation

$$\eta_m = \psi(\eta'), \ \ \eta' \in \mathbb{R}^2,$$

where $\psi(\eta')$ satisfies the condition

$$|\psi'(\eta')| \leq M < \infty, \quad = const.$$

We put

$$q = \max_D \psi'(\eta'), \ \ l = \max_D \sqrt{1 + \psi'^2(\eta')}.$$

Theorem 4. *Let* $W(\eta) \in S_\rho(\Omega)$ *satisfies in the boundary condition Equation (25), and on a smooth surface* Σ *the inequality*

$$|W(\eta)| \leq \delta, \ \ 0 < \delta < 1. \tag{39}$$

Then the following estimates will be valid

$$|W(\zeta)| \leq K\_\rho(\lambda, \zeta)\sigma M\text{^}(1 - \zeta\_3/q)\ \delta\text{^}(\zeta\_3/q), \ \ \sigma > 1, \ \ \zeta \in \Omega. \tag{40}$$

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j}\right| \leq K_\rho(\lambda, \zeta)\sigma M^{1-\frac{\zeta_3}{q}}\delta^{\frac{\zeta_3}{q}}, \ \ \sigma > 1, \ \ \zeta \in \Omega, \quad j = \overline{1,3}. \tag{41}$$

Proof. Let us first evaluate the validity of Equation (40). Based on the integral representation Equation (24), we obtain

$$W(\zeta) = \int_\Sigma L_\sigma(\eta, \zeta; \lambda)W(\eta)ds_\eta + \int_D L_\sigma(\eta, \zeta; \lambda))W(\eta)ds_\eta, \ \ \zeta \in \Omega. \tag{42}$$

We estimate the following

$$|W(\zeta)| \leq \left|\int_\Sigma L_\sigma(\eta, \zeta; \lambda)W(\eta)ds_\eta\right| + \left|\int_D L_\sigma(\eta, \zeta; \lambda)W(\eta)ds_\eta\right|, \ \ \zeta \in \Omega. \tag{43}$$

Thanks to Equation (39), we first estimate the first integral Equation (43).

$$\left|\int_\Sigma L_\sigma(\eta, \zeta; \lambda)W(\eta)ds_\eta\right| \leq \int_\Sigma |L_\sigma(\eta, \zeta; \lambda)||W(\eta)|ds_\eta \leq \leq \delta \int_\Sigma |L_\sigma(\eta, \zeta; \lambda)|ds_\eta, \ \ \zeta \in \Omega. \tag{44}$$

Here we will appreciate $\int_\Sigma |\Psi_\sigma(\eta, \zeta; \lambda)|ds_\eta$, $\int_\Sigma \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_j}\right| ds_\eta$, $j = \overline{1,2}$ and $\int_\Sigma \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_3}\right| ds_\eta$ on a Σ.

Based on equality (30), we obtain the estimate

$$\int_\Sigma |\Psi_\sigma(\eta, \zeta; \lambda)|ds_\eta \leq K_\rho(\lambda, \zeta)\sigma e^{\sigma(q-\zeta_3)}, \ \ \sigma > 1, \ \ \zeta \in \Omega. \tag{45}$$

And now, to estimate the second integral, based on Equations (30) and (32), we obtain, respectively,

$$\int_\Sigma \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_j}\right| ds_\eta \le K_\rho(\lambda,\zeta)\sigma e^{\sigma(q-\zeta_3)}, \ \ \sigma > 1, \ \ \zeta \in \Omega, \ \ j = \overline{1,2}. \tag{46}$$

When estimating the integral $\int_\Sigma \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_3}\right| ds_\eta$, we take Equation (30) into account and obtain

$$\int_\Sigma \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_3}\right| ds_\eta \le K_\rho(\lambda,\zeta)\sigma e^{\sigma(q-\zeta_3)}, \ \ \sigma > 1, \ \ \zeta \in \Omega. \tag{47}$$

From the obtained estimates Equations (45)–(47), and also on the basis of Equation (44), we obtain

$$\left|\int_\Sigma L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta\right| \le K_\rho(\lambda,\zeta)\sigma\delta e^{\sigma(q-\zeta_3)}, \ \ \sigma > 1, \ \ \zeta \in \Omega. \tag{48}$$

The following is known

$$\left|\int_D L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta\right| \le K_\rho(\lambda,\zeta)\sigma M e^{-\sigma\zeta_3}, \ \ \sigma > 1, \ \ \zeta \in \Omega. \tag{49}$$

Now taking into account Equations (48) and (49) and using Equation (43), we have

$$|W(\zeta)| \le (K\_\rho(\lambda,\zeta)\sigma)/2(\delta e\^\sigma q + M)e\^(-\sigma\zeta\_3), \ \ \sigma > 1, \ \ \zeta \in \Omega. \tag{50}$$

Choosing σ from the equality

$$\sigma = \frac{1}{q}\ln\frac{M}{\delta}, \tag{51}$$

we will obtain proof Equation (41).

Now it remains to prove Equation (41). Here we first find the partial derivative of Equation (24) with respect to the variable $\zeta_j, \ j = \overline{1,3}$:

$$\begin{aligned}\frac{\partial W(\zeta)}{\partial \zeta_j} &= \int_\Sigma \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta)ds_\eta + \int_D \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta)ds_\eta = \\ &= \frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} + \int_D \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta)ds_\eta, \ \ \zeta \in \Omega, \ \ j = \overline{1,3}.\end{aligned} \tag{52}$$

Where

$$\frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} = \int_\Sigma \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta)ds_\eta. \tag{53}$$

We estimate the following

$$\begin{aligned}\left|\frac{\partial W(\zeta)}{\partial \zeta_j}\right| &\le \left|\int_\Sigma \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta)ds_\eta\right| + \\ &+ \left|\int_D \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta)ds_\eta\right| \le \left|\frac{\partial W_\sigma(\zeta)}{\partial \zeta_j}\right| + \\ &+ \left|\int_D \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta)ds_\eta\right|, \ \ \zeta \in \Omega, \ \ j = \overline{1,3}.\end{aligned} \tag{54}$$

Based on Equation (40), we will estimate the first integral Equation (54).

$$\left|\int_{\Sigma}\frac{\partial L_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}W(\eta)ds_{\eta}\right| \leq \int_{\Sigma}\left|\frac{\partial L_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}\right||W(\eta)|ds_{\eta} \leq$$
$$\leq \delta\int_{\Sigma}\left|\frac{\partial L_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}\right|ds_{\eta}, \quad \zeta\in\Omega, \quad j=\overline{1,3}. \tag{55}$$

To prove Equation (55), we estimate the integrals $\int_{\Sigma}\left|\frac{\partial \Psi_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}\right|ds_y$, $j=\overline{1,2}$ and $\int_{\Sigma}\left|\frac{\partial \Psi_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_3}\right|ds_{\eta}$ on a Σ.
Based on equalities Equations (30) and (36), we finally obtain

$$\int_{\Sigma}\left|\frac{\partial \Psi_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}\right|ds_{\eta} \leq K_{\rho}(\lambda,\zeta)\sigma e^{\sigma(q-\zeta_3)}, \quad \sigma>1, \ \zeta\in\Omega, \ j=\overline{1,2}. \tag{56}$$

Now let's move on to estimating $\int_{\Sigma}\left|\frac{\partial \Psi_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_3}\right|ds_{\eta}$.
Based on Equation (30), we finally obtain

$$\int_{\Sigma}\left|\frac{\partial \Psi_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_3}\right|ds_{y} \leq K_{\rho}(\lambda,\zeta)\sigma e^{\sigma(q-\zeta_3)}, \quad \sigma>1, \ \zeta\in\Omega, \tag{57}$$

From the already obtained estimates Equations (56) and (57), and also on the basis of Equation (55), we obtain

$$\left|\int_{\Sigma}\frac{\partial L_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}W(\eta)ds_{\eta}\right| \leq K_{\rho}(\lambda,\zeta)\sigma\delta e^{\sigma(q-\zeta_3)}, \ \sigma>1, \ \zeta\in\Omega,$$
$$j=\overline{1,3}. \tag{58}$$

We received the following

$$\left|\int_{D}\frac{\partial L_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}W(\eta)ds_{\eta}\right| \leq K_{\rho}(\lambda,\zeta)\sigma M e^{-\sigma\zeta_3}, \ \sigma>1, \ \zeta\in\Omega,$$
$$j=\overline{1,3}. \tag{59}$$

From the estimates obtained above Equations (58)–(59), based on Equation (54), we have as a result

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j}\right| \leq \frac{K_{\rho}(\lambda,\zeta)\sigma}{2}(\delta e^{\sigma q}+M)e^{-\sigma\zeta_3}, \ \sigma>1, \ \zeta\in\Omega,$$
$$j=\overline{1,3}. \tag{60}$$

In the last estimate, choosing σ from Equation (51), we finally obtain the validity of Equation (41). □

Assume that $W(\eta)\in S_{\rho}(\Omega)$ is defined on Σ and $f_{\delta}(\eta)$ is its approximation with an error $0<\delta<1$ in this case

$$\max_{\Sigma}|W(\eta)-f_{\delta}(\eta)|\leq\delta. \tag{61}$$

We put

$$W_{\sigma(\delta)}(\zeta)=\int_{\Sigma}L_{\sigma}(\eta,\zeta;\lambda)f_{\delta}(\eta)ds_{\eta}, \quad \zeta\in\Omega. \tag{62}$$

Theorem 5. *Let $W(\eta)\in S_{\rho}(\Omega)$ on the part of the plane $\eta_3=0$ satisfies in the*

condition Equation (25).
In this case, the following are true:

$$W_{\sigma(\delta)}(\zeta) = \int_{\Sigma} L_{\sigma}(\eta, \zeta; \lambda) f_{\delta}(\eta) ds_{\eta}, \ \ \zeta \in \Omega. \tag{63}$$

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j} - \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\right| \le K_{\rho}(\lambda, \zeta) \sigma M^{1-\frac{\zeta_3}{q}} \delta^{\frac{\zeta_3}{q}}, \ \ \sigma > 1, \ \ \zeta \in \Omega, \\ j = \overline{1,3}. \tag{64}$$

Proof. Based on the integral representations Equations (24) and (62), we will have

$$W(\zeta) - W_{\sigma(\delta)}(\zeta) = \int_{\partial\Omega} L_{\sigma}(\eta, \zeta; \lambda) W(\eta) ds_{\eta} - \\ - \int_{\Sigma} L_{\sigma}(\eta, \zeta; \lambda) f_{\delta}(\eta) ds_{\eta} = \int_{\Sigma} L_{\sigma}(\eta, \zeta; \lambda) W(\eta) ds_{\eta} + \\ + \int_{D} L_{\sigma}(\eta, \zeta; \lambda) W(\eta) ds_{\eta} - \int_{\Sigma} L_{\sigma}(\eta, \zeta; \lambda) f_{\delta}(\eta) ds_{\eta} = \\ = \int_{\Sigma} L_{\sigma}(\eta, \zeta; \lambda) \{W(\eta) - f_{\delta}(\eta)\} ds_{\eta} + \int_{D} L_{\sigma}(\eta, \zeta; \lambda) W(\eta) ds_{\eta}.$$

and

$$\frac{\partial W(\zeta)}{\partial \zeta_j} - \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j} = \int_{\partial\Omega} \frac{\partial L_{\sigma}(\eta, \zeta; \lambda)}{\partial \zeta_j} W(\eta) ds_{\eta} - \\ - \int_{\Sigma} \frac{\partial L_{\sigma}(\eta, \zeta; \lambda)}{\partial \zeta_j} f_{\delta}(y) ds_{y} = \int_{S} \frac{\partial N_{\sigma}(y, x; \lambda)}{\partial x_j} U(y) ds_{y} + \\ + \int_{D} \frac{\partial L_{\sigma}(\eta, \zeta; \lambda)}{\partial \zeta_j} W(\eta) ds_{\eta} - \int_{\Sigma} \frac{\partial L_{\sigma}(\eta, \zeta; \lambda)}{\partial \zeta_j} f_{\delta}(\eta) ds_{\eta} = \\ = \int_{\Sigma} \frac{\partial L_{\sigma}(\eta, \zeta; \lambda)}{\partial \zeta_j} \{W(\eta) - f_{\delta}(\eta)\} ds_{\eta} + \int_{D} \frac{\partial L_{\sigma}(\eta, \zeta; \lambda)}{\partial \zeta_j} W(\eta) ds_{\eta}, \\ j = \overline{1,3}.$$

Based on Equations (25) and (61), we will further estimate the following:

$$\left|W(\zeta) - W_{\sigma(\delta)}(\zeta)\right| = \left|\int_{\Sigma} L_{\sigma}(\eta, \zeta; \lambda) \{W(\eta) - f_{\delta}(\eta)\} ds_{\eta}\right| + \\ + \left|\int_{D} L_{\sigma}(\eta, \zeta; \lambda) W(\eta) ds_{\eta}\right| \le \int_{\Sigma} |L_{\sigma}(\eta, \zeta; \lambda)| |\{W(\eta) - f_{\delta}(\eta)\}| ds_{\eta} + \\ + \int_{D} |L_{\sigma}(\eta, \zeta; \lambda)| |W(\eta)| ds_{\eta} \le \delta \int_{\Sigma} |L_{\sigma}(\eta, \zeta; \lambda)| ds_{\eta} + \\ + M \int_{D} |L_{\sigma}(\eta, \zeta; \lambda)| ds_{\eta}.$$

and

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j}-\frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\right|=\left|\int_{\sigma}\frac{\partial L_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}\{W(\eta)-f_{\delta}(\eta)\}ds_{\eta}\right|+$$
$$+\left|\int_{D}\frac{\partial L_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}W(\eta)ds_{\eta}\right|\leq\int_{\Sigma}\left|\frac{\partial L_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}\right||\{W(\eta)-f_{\delta}(\eta)\}|ds_{\eta}+$$
$$+\int_{D}\left|\frac{\partial L_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}\right||W(\eta)|ds_{\eta}\leq\delta\int_{\Sigma}\left|\frac{\partial L_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}\right|ds_{\eta}+$$
$$+M\int_{D}\left|\frac{\partial L_{\sigma}(\eta,\zeta;\lambda)}{\partial \zeta_j}\right|ds_{\eta},\quad j=\overline{1,3}.$$

To prove this theorem, we will use the already known results of Theorems 3 and 4.

$$\left|W(\zeta)-W_{\sigma(\delta)}(\zeta)\right|\leq\frac{K_{\rho}(\lambda,\zeta)\sigma}{2}(\delta e^{\sigma q}+M)e^{-\sigma\zeta_3}.$$

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j}-\frac{W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\right|\leq\frac{K_{\rho}(\lambda,\zeta)\sigma}{2}(\delta e^{\sigma q}+M)e^{-\sigma\zeta_3},\ \ j=\overline{1,3}.$$

In the last estimates, choosing σ from Equation (51), we will fully prove the validity of estimates Equations (63) and (64). □

Corollary 3. *For each $\zeta\in\Omega$ the following corresponding limit equalities hold:*

$$\lim_{\delta\to 0}W_{\sigma(\delta)}(\zeta)=W(\zeta),\quad \lim_{\delta\to 0}\frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}=\frac{\partial W(\zeta)}{\partial \zeta_j},\ \ j=\overline{1,3}.$$

Corollary 4. *If $\zeta\in\overline{\Omega}_{\varepsilon}$, then the families of functions $\{W_{\sigma(\delta)}(\zeta)\}$ and $\left\{\frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\right\}$ are convergent uniformly for $\delta\to 0$, i.e.:*

$$W_{\sigma(\delta)}(\zeta)\rightrightarrows W(\zeta),\quad \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\rightrightarrows\frac{\partial W(\zeta)}{\partial \zeta_j},\ \ j=\overline{1,3}.$$

4. Conclusions

In this paper, we have found an approximate solution to the problem based on the properties of the Carleman matrix. If the Carleman matrix is known, then it is no longer difficult to find a regularized solution in explicit form. In this case, we have that the solution to the problem exists and is continuously differentiable in a closed region with exactly specified Cauchy data.

We note that for solving applicable problems, the approximate values of $W(\zeta)$ and $\frac{\partial W(\zeta)}{\partial \zeta_j},\ \zeta\in\Omega,\ j=\overline{1,3}$ should be found.

As a result, we constructed a family of vector functions $W(\zeta,f_{\delta})=W_{\sigma(\delta)}(\zeta)$ and $\frac{\partial w(\zeta,f_{\delta})}{\partial \zeta_j}=\frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j},\ j=\overline{1,3}$, which depend on the parameter σ. It is additionally proved that under specific conditions and a special choice of the parameter $\sigma=\sigma(\delta)$, at $\delta\to 0$, the family $W_{\sigma(\delta)}(\zeta)$ and $\frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}$ are convergent to a solution $W(\zeta)$ and its derivative $\frac{\partial W(\zeta)}{\partial \zeta_j},\zeta\in\Omega$ at point $\zeta\in\Omega$. Here we will call $W_{\sigma(\delta)}(\zeta)$ and $\frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}$ the regularized solution of the problems Equations (1) and

(2).

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