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Brief Report

Approximation results of Phillips type operators including exponential function

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Abstract: The current article deals with a study on some moderation of the Phillips operators, including constant and exponential functions. Here, we derive the moments applying the notion of moment-generating function for the well-known Phillips operators. The authors also establish uniform convergence estimates for the improved form of these operators. Additionally, some direct estimates involving the asymptotic-type results are discussed.

Keywords: Phillips operators; approximation; exponential functions; moments; linear positive operators
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1. Introduction

Approximation theory is one of the important subjects that is frequently used by the mathematical and scientific fraternity. It is divided into many fields. Here we are dealing with positive linear operators that play a key role in the field of approximation theory. A few positive linear operators, e.g., Bernstein operators, are described within finite intervals, but there are many such operators that are interpreted in infinite intervals, such as Baskakov [1] operators, which are given as below.

$$B_n(f;x) = \frac{1}{(1+x)^n}\sum_{v=0}^{\infty} f\left(\frac{v}{n}\right) C(n+v-1,v)\frac{x^v}{(1+x)^v}$$

where $n \in \mathbb{N}$ and $f \in C[0,\infty)$.

In recent years, several operators were appropriately moderated, preserving the test functions that appeared in this field, and many modifications have been carried out regarding this matter to obtain a better approximation. One may see a few of the results in the related direction [2–8].

Phillips operators, named after the name of great mathematician Phillips [9], are defined as:

$$P_n(f;x) = n\sum_{v=1}^{\infty} e^{-nx}\frac{(nx)^v}{v!}\int_0^{\infty} e^{-nt}\frac{(nt)^{v-1}}{(v-1)!}f(t)\mathrm{d}t + e^{-nx}f(0) \tag{1}$$

The above Equation (1) preserves both constant and linear functions. Approximation properties of Phillips-type operators have been discussed by many researchers in [10–16]. Inspired by the work of King [17], in the year 2010, Gupta [18] introduced the moderated Phillips operators, which preserve e_2. The modified formation of operators shows better approximation in comparison to normal Phillips operators. Through normal calculation and computation, we have:

$$\mu_x(\theta) = P_n\left(e^{\theta t}; x\right) = e^{\left(\frac{nx\theta}{n-\theta}\right)} \tag{2}$$

Which is the moment-originating function of the operators P_n and it will be useful in finding the moments of the Phillips operators.

The moments are given as:

$$P_n\left(e^{\theta t}; x\right) = 1 + x\theta + \left(x^2 + \frac{2x}{n}\right)\frac{\theta^2}{2!} + \left(\frac{6x + 6nx^2 + n^2x^3}{n^2}\right)\frac{\theta^3}{3!} + \left(\frac{24x + 36nx^2 + 12n^2x^3 + n^3x^4}{n^3}\right)\frac{\theta^4}{4!} + \left(\frac{120x + 240nx^2 + 120n^2x^3 + 20n^3x^4 + n^4x^5}{n^4}\right)\frac{\theta^5}{5!} + \left(\frac{720x + 1800nx^2 + 1200n^2x^3 + 300n^3x^4 + 30n^4x^5 + n^5x^6}{n^5}\right) \times \frac{\theta^6}{6!} + O(\theta^7)$$

Hence, it can be seen that:

$$M_{n,r}^{P_n}(x) = \left[\frac{\mathrm{d}^r}{\mathrm{d}\theta^r} P_n\left(e^{\theta t}; x\right)\right]_{\theta=0} = \left[\frac{\mathrm{d}^r}{\mathrm{d}\theta^r} e^{\left(\frac{nx\theta}{n-\theta}\right)}\right]_{\theta=0}, \tag{3}$$

where $M_{n,r}^{P_n}(x) = P_n(e_k; x), e_k(t) = t^k, k = 0,1,2, \ldots$

Acar et al. [19] proposed a modification of linear positive operators to reproduce the function e^{2A} .

The current article is organized as follows:

In the starting section, we obtain modified Phillips operators preserving exponential functions. The authors study the quantifiable estimate for the Phillips operators, preserving the function e^{-x}. Section 3 deals with the preservation for e^{Ax}, where *A* is real.

In section 2 and section 3, the authors have shown that moderated operators furnish better approximations than the standard Phillips operators.

2. Results for the preservation of e^{-x}

In this section, we present the preservation of e^{-x} and also establish some lemmas.

For the preservation of the function, we have:

$$S_n(f; x) = n \sum_{v=1}^{\infty} e^{-n\alpha_n(x)} \frac{\left(n\alpha_n(x)\right)^v}{v!} \int_0^{\infty} e^{-nt} \frac{(nt)^{v-1}}{(v-1)!} f(t)\mathrm{d}t + e^{-n\alpha_n(x)} f(0) \tag{4}$$

Let the Equation (4) conserve e^{-x} means $S_n(e^{-t}; x) = e^{-x}$

Hence, we obtain $e^{-x} = e^{\frac{-n\alpha_n(x)}{n+1}}$ suggesting that:

$$\alpha_n(x) = \frac{x(n+1)}{n} \tag{5}$$

We can also write the Equation (4) as follows:

$$S_n(f;x) = n\sum_{v=1}^{\infty} e^{-x(n+1)} \frac{(x(n+1))^v}{v!} \int_0^{\infty} e^{-nt} \frac{(nt)^{v-1}}{(v-1)!} f(t)\mathrm{d}t + e^{-x(n+1)} f(0)$$

Lemma 1. *Using normal computation, we have:*

$$S_n(e^{At};x) = e^{\frac{Ax(n+1)}{n-A}}$$

Lemma 2. *For the operators in Equation (4), if* $T_{n,m}(x) = S_n(e_m;x)$ *with* $e_k(t) = t^k, k = 0,1,2,\dots$ *then using Equation (3), we get:*

$$S_n(e_0;\, x) = 1$$

$$S_n(e_1;\, x) = \alpha_n(x)$$

$$S_n(e_2;x) = \alpha_n^2(x) + \frac{2\alpha_n(x)}{n}$$

$$S_n(e_3;x) = \alpha_n^3(x) + \frac{6\alpha_n^2(x)}{n} + \frac{6\alpha_n(x)}{n^2}$$

$$S_n(e_4;x) = \alpha_n^4(x) + \frac{12\alpha_n^3(x)}{n} + \frac{36\alpha_n^2(x)}{n^2} + \frac{24\alpha_n(x)}{n^3}.$$

Lemma 3. *Let* $M_{n,k}(x) = S_n\left((t-x)^k;x\right), k = 0,1,2,\dots$ *then by making the application of Lemma 2, we get:*

$$M_{n,0}(x) = 1$$

$$M_{n,1}(x) = \alpha_n(x) - x = \frac{x}{n}$$

$$M_{n,2}(x) = (\alpha_n(x) - x)^2 + \frac{2\alpha_n(x)}{n} = \frac{x^2}{n^2} + \frac{2x}{n} + \frac{2x}{n^2}$$

$$M_{n,4}(x) = (\alpha_n(x) - x)^4 + \frac{12\alpha_n^3(x)}{n} + \frac{36\alpha_n^2(x)}{n^2} + \frac{24\alpha_n(x)}{n^3} - \frac{24x\alpha_n^2(x)}{n} - \frac{24x\alpha_n(x)}{n^2} + \frac{12x^2\alpha_n(x)}{n}$$

From Equation (5), it follows that:

$$\lim_{n\to\infty} n(\alpha_n(x) - x) = x$$

$$\lim_{n\to\infty} n\left((\alpha_n(x) - x)^2 + \frac{2\alpha_n(x)}{n}\right) = 2x \tag{6}$$

Let $C^*[0,\infty)$ be the subclass of real-valued continual functions with a finite limit at infinity endowed with the uniform norm. Boyanov [20] established the uniform convergence of a sequence of linear positive operators.

Proposition 1. *Holhos [21] observes a sequence of positive linear operators:*

$$Q_n\colon C^*[0,\infty) \to C^*[0,\infty)$$

and set:

$$\|Q_n e_0 - 1\|_{[0,\infty)} = \alpha_n$$
$$\|Q_n(e^{-t}) - e^{-x}\|_{[0,\infty)} = \beta_n$$
$$\|Q_n(e^{-2t}) - e^{-2x}\|_{[0,\infty)} = \gamma_{n.}$$

If $\alpha_n, \beta_n, \gamma_n$ approaches to 0 as $n \to \infty$, then:

$$\|Q_n f - f\|_{[0,\infty)} \le \alpha_n \parallel f \parallel_{[0,\infty)} + (2 + \alpha_n)\omega^*\left(f, \sqrt{\alpha_n + 2\beta_n + \gamma_n}\right).$$

The modulus of continuity is described as:

$$\omega^*(f, \delta) := \sup_{\substack{|e^{-x} - e^{-t}| \le \delta \\ x,t > 0}} |f(t) - f(x)|.$$

Theorem 1. *For $f \in C^*[0, \infty)$, we have:*

$$\|S_n f - f\|_{[0,\infty)} \le 2\omega^*\left(f, \sqrt{\gamma_n}\right),$$

where:

$$\gamma_n = \|S_n(e^{-2t}) - e^{-2x}\|_{[0,\infty)} = \left(1 - \frac{1}{n+2}\right)^{n+2} (n+1)^{-1}.$$

Proof. The operators S_n conserve constant functions and also e^{-x}; therefore $\alpha_n = \beta_n = 0$. In order to compute γ_n, in view of Lemma 1, we have:

$$S_n(e^{-2t}; x) = e^{\frac{-2x(n+1)}{n+2}} = e^{-2x} \cdot e^{\frac{2x}{n+2}}.$$

Considering the function:

$$f_n(x) = S_n(e^{-2t}, x) - e^{-2x}, x \ge 0.$$

For a function which is positive, $f_n(0) = 0$, $\lim_{x \to +\infty} f_n(x) = 0$ and it has its maxima at point:

$$x_n = \frac{n+2}{2} \log\left(\frac{n+2}{n+1}\right).$$

Hence, we have:

$$\gamma_n = \|f_n\|_{[0,\infty)} = f_n(x_n) = \left(1 - \frac{1}{n+2}\right)^{n+2} (n+1)^{-1}.$$

This carried out for the proof of the theorem. □

Remark 1. *For the operators defined in Equation (1), according to Proposition 1, we have:*

$$\|P_n f - f\|_{[0,\infty)} \le 2\omega^*\left(f, \sqrt{2\beta_n + \gamma_n}\right),$$

using Equation (2), we have:

$$\beta_n = \|P_n(e^{-t}) - e^{-x}\|_{[0,\infty)} = \left(1 - \frac{1}{n+1}\right)^{n+1} n^{-1},$$

and

$$\gamma_n = \|P_n(e^{-2t}) - e^{-2x}\|_{[0,\infty)} = \left(1 - \frac{2}{n+2}\right)^{(n+2)/2} 2n^{-1}.$$

Hence Proposition 1, gives finer approximation results for the modified Phillips operators S_n than the regular Phillips operators P_n.

Remark 2. According to Equation (4) it can be seen:

$$S_n(f;\ x) = P_n\big(f; \alpha_n(x)\big). \tag{7}$$

We also represent S_n as in Theorem 2.

Theorem 2. *For all $f \in C[0,\infty)$, we set:*

$$S_n(f(t\,; x) = P_{n+1}\left(f\left(\frac{(n+1)t}{n}\right); x\right). \tag{8}$$

Proof. On substituting $t = \frac{n+1}{n}v$, the proof follows from Equation (7).

We describe a function $h(t)$ as:

$$h(t) = f\left(\frac{(n+1)t}{n}\right), f \in C_B[0,\infty) \tag{9}$$

where $C_B[0,\infty)$ indicates the subinterval of all bounded continuous function on $[0,\infty)$.

Hence, we get:

$$S_n(f,x) - f(x) = P_{n+1}(h,x) - h(x) + h(x) - f(x).$$

So,

$$\|S_n f - f\|_{[0,\infty)} \le \|P_{n+1}h - h\|_{[0,\infty)} + \|\ h - f\ \|_{[0,\infty)} \tag{10}$$

According to Heilmann and Tachev [22], for every $f \in C_B[0,\infty)$

$$\|P_n f - f\|_{[0,\infty)} \le 2K_\varphi^2(f; n^{-1}) \le C\omega_\varphi^2\left(f; n^{-\frac{1}{2}}\right). \tag{11}$$

We see that the Peetre's K-functional and the Ditzian-Totik modulus of continuity follow from Heilmann and Tachev [22].

It is clear that:

$$\|\ h - f\ \|_{[0,\infty)} \le \omega(f; n^{-1}).\ \|P_n f - f\|_{[0,\infty)} \le 2K_\varphi^2(f; n^{-1}) \le C\omega_\varphi^2\left(f; n^{-\frac{1}{2}}\right) \tag{12}$$

Now, from Equations (10)–(12), we get the proof. □

Theorem 3. *For every $f \in C_B[0,\infty)$ and n is any positive integer, inequality (13) holds.*

$$\|S_n f - f\|_{[0,\infty)} \le C\omega_\varphi^2\big(f, n^{-1/2}\big) + \omega(f, n^{-1}). \tag{13}$$

The asymptotic-type outcome for the operators S_n is showed in the following.

Theorem 4. *Consider $f, f'' \in C^*[0,\infty)$, now for any $x \in [0,\infty)$, we obtain:*

$$|n[S_n(f;x) - f(x)] - x[f'(x) + f''(x)]| \le \frac{x(x+2)}{2n}|f''(x)|$$

$$+2\left[\frac{x^2}{n} + 2x + \frac{2x}{n} + k_n(x)\right]\cdot \omega^*\big(f''(x), n^{-1/2}\big),$$

where:

$$k_n(x) = n^2\big[S_n((e^{-x} - e^{-t})^4, x) \cdot M_{n,4}(x)\big]^{1/2}.$$

Proof. Using Taylor's expansion, we derive:

$$f(t) = f(x) + (t - x)f'(x) + \frac{1}{2}(t - x)^2 + g(t,x)(t - x)^2 \tag{14}$$

Noting that:

$$g(t,x) = \frac{f''(\eta) - f''(x)}{2}.$$

η lies between x and t. On applying the operator S_n to both the sides of Equation (14), we obtain:

$$\left|S_n(f;x) - f(x) - M_{n,1}(x)f'(x) - \frac{1}{2}M_{n,2}(x)f''(x)\right| \leq |S_n(g(t,x)(t - x)^2;x)|.$$

By Lemma 3, authors get:

$$|n[S_n(f;x) - f(x)] - x[f'(x) + f''(x)]|$$

$$\leq \left|nM_{n,1}(x) - x\right||f'(x)| + \frac{1}{2}\left|nM_{n,2}(x) - 2x\right||f''(x)|$$

$$+|nS_n(g(t,x)(t - x)^2;x)|$$

$$\leq \frac{x(x + 2)}{2n} \cdot |f''(x)| + |nS_n(g(t,x)(t - x)^2;x)|.$$

For the proof of the above theorem, we need to estimate $|nS_n(g(t,x)(t - x)^2;x)|$. Using the inequality:

$$|f(t) - f(x)| \leq \left(1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2}\right)\omega^*(f,\delta), \delta > 0,$$

we obtain:

$$|g(t,x)| \leq \left(1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2}\right)\omega^*(f'',\delta).$$

For $|e^{-x} - e^{-t}| \leq \delta$, we have $|g(t,x)| \leq 2\omega^*(f'',\delta)$.
When $|e^{-x} - e^{-t}| > \delta$, then:

$$|g(t,x)| < 2\frac{(e^{-x} - e^{-t})^2}{\delta^2}\omega^*(f'',\delta).$$

Therefore,

$$|g(t,x)| \leq 2\left(1 + \frac{(e^{-x} - e^{-t})^2}{\delta^2}\omega^*(f'',\delta)\right).$$

Applying above result and inequality of Cauchy-Schwarz, we have for $\delta = n^{-1/2}$.

$$nS_n(|g(t,x)|(t - x)^2;x) \leq 2n\omega^*(f''(x),\delta)M_{n,2}(x)$$

$$+\frac{2n}{\delta^2} \mid \omega^*(f''(x),\delta)[S_n((e^{-x} - e^{-t})^4;x)]^{1/2} \cdot \big[M_{n,4}(x)\big]^{1/2}$$

$$= 2\left[\frac{x^2}{n} + 2x + \frac{2x}{n} + k_n(x)\right]\omega^*\big(f''(x), n^{-1/2}\big).$$

Where, $k_n(x) = n^2\big[S_n((e^{-x} - e^{-t})^4, x) \cdot M_{n,4}(x)\big]^{1/2}$. □

Corollary 1. Suppose $f, f'' \in C^*[0,\infty)$, *so for* $x \in [0,\infty)$, we get:

$$\lim_{n\to\infty} n[S_n(f,x) - f(x)] = x[f'(x) + f''(x)].$$

3. Results for e^{Ax}, where *A* is real

In this section, we achieve a moderation of the Phillips operator for any real *A*, where a copy of e^{At} (an exponential function) is produced. From Lemma 1, we observe that:

$$S_n(e^{At};x) = e^{\frac{An \quad n(x)}{n-A}} = e^{Ax}.$$

For,

$\alpha_n(x) = x\left(\frac{n-A}{n}\right) = x\left(1 - \frac{A}{n}\right)$, considering the operators

$$S_n^A(f;x) = n\sum_{v=1}^{\infty} e^{-x(n-A)}\frac{(x(n-A))^v}{v!}\int_0^{\infty} e^{-nt}\frac{(nt)^{v-1}}{(v-1)!}f(t)\mathrm{d}t + e^{-x(n+1)}f(0). \quad (15)$$

Hence, the moment-generating function of above Equation (15), is given as:

$$\mu_x(\theta) = e^{\left(\frac{(n-A)\theta x}{n-\theta}\right)}. \quad (16)$$

Now from Lemma 2, we have:

$$S_n^A(e_0;\ x) = 1$$

$$S_n^A(e_1;\ x) = x\left(1 - \frac{A}{n}\right)$$

$$S_n^A(e_2;\ x) = x\left(1 - \frac{A}{n}\right)\left[x\left(1 - \frac{A}{n}\right) + \frac{2}{n}\right].$$

From Lemma 3, we get the following representation:

$$M_{n,1}^{S_n^A}(x) = -\frac{Ax}{n} \quad (17)$$

$$M_{n,2}^{S_n^A}(x) = \frac{x^2A^2}{n^2} - \frac{2xA}{n^2} + \frac{2x}{n} \quad (18)$$

$$M_{n,4}^{S_n^A}(x) = \frac{x^4A^4}{n^4} + \frac{12x^3A^2}{n^3}\left(1 - \frac{A}{n}\right) + \frac{36x^2}{n^2}\left(1 - \frac{A}{n}\right)^2 + \frac{24x}{n^2}\left(1 - \frac{A}{n}\right)\left(\frac{1}{n} - x\right) \quad (19)$$

Therefore, by Equations (18) and (19), we get:

$$\frac{M_{n,4}^{S_n^A}(x)}{M_{n,2}^{S_n^A}(x)} = \frac{\frac{x^3A^4}{n^3} + \frac{12\ \ ^2A^2}{n^2}\left(1 - \frac{A}{n}\right) + \frac{36}{n}\left(1 - \frac{A}{n}\right)^2 + \frac{24}{n}\left(1 - \frac{A}{n}\right)\left(\frac{1}{n} - x\right)}{\frac{xA^2}{n} - \frac{2A}{n} + 2}.$$

Therefore, for any for fixed $x \in [0,\infty)$, when $n \to \infty$, we have the following result:

$\frac{M_{n,4}^{S_n^A}(x)}{M_{n,2}^{S_n^A}(x)} \to 0$, having order of convergence as $O\left(\frac{1}{n}\right)$.

Lemma 4. *For any real A, we have:*

$$S_n^A(e^{At}(t-x)^2;x) = \frac{x^2A^2+2nx}{(n-A)^2}e^{Ax}.$$

Proof. For any $A \in \mathbb{R}$, let a function $f(t) = e^{At}$, then:

$$n\int_0^\infty e^{-nt}\frac{(nt)^{v-1}}{(v-1)!}e^{At}t^k\mathrm{d}t = \frac{n^v(v+k-1)!}{(v-1)!\,(n-A)^{v+k}}.$$

Hence, we have:

$$\begin{aligned} S_n^A(e^{At}t;x) &= \sum_{v=1}^{\infty} e^{-x(n-A)}\frac{(x(n-A))^v}{v!}\frac{n^v v!}{(v-1)!\,(n-A)^{v+1}} \\ &= \frac{e^{-x(n-A)}}{n-A}\sum_{v=1}^{\infty}\frac{(nx)^v}{(v-1)!} \\ &= \frac{nx\cdot e^{-x(n-A)}}{n-A}\sum_{v=0}^{\infty}\frac{(nx)^v}{v!} \\ &= \frac{nx\cdot e^{Ax}}{n-A}. \end{aligned}$$

Finally, we have:

$$\begin{aligned} S_n^A(e^{At}t^2;x) &= \sum_{v=1}^{\infty} e^{-x(n-A)}\frac{(x(n-A))^v}{v!}\frac{n^v(v+1)!}{(v-1)!\,(n-A)^{v+2}} \\ &= \frac{e^{-x(n-A)}}{(n-A)^2}\sum_{v=1}^{\infty}\frac{(nx)^v}{(v-1)!}(v+1) \\ &= \frac{e^{-x(n-A)}}{(n-A)^2}\left[\sum_{v=2}^{\infty}\frac{(nx)^v}{(v-2)!} + 2\sum_{v=1}^{\infty}\frac{(nx)^v}{(v-1)!}\right] \\ &= \frac{e^{-x(n-A)}}{(n-A)^2}\left[n^2x^2\sum_{v=0}^{\infty}\frac{(nx)^v}{v!} + 2nx\sum_{v=0}^{\infty}\frac{(nx)^v}{v!}\right] \\ &= \frac{e^{Ax}}{(n-A)^2}[n^2x^2+2nx]. \end{aligned}$$

Hence:

$$\begin{aligned} S_n^A(e^{At}(t-x)^2;x) &= S_n^A(e^{At}t^2,x) - 2xS_n^A(e^{At};,x) + x^2S_n^A(e^{At};x) \\ &= \frac{e^{Ax}}{(n-A)^2}[n^2x^2+2nx] - \frac{2nx^2e^{Ax}}{n-A} + x^2e^{Ax} \end{aligned}$$

$$= e^{Ax}\left[\frac{n^2x^2 + 2nx}{(n-A)^2} - \frac{2nx^2}{n-A} + x^2\right]$$

$$= \frac{x^2A^2 + 2nx}{(n-A)^2}e^{Ax}.$$

Thus,

Exponential growth for continuous functions on $[0, \infty)$ is given by:

$$\| f \|_A := \sup_{x\in[0,\infty)} |f(x)e^{-Ax}| < \infty, A > 0.$$

Ditzian [10] considered the modulus of continuity of second order as follows:

$$\omega_2(f, \delta, A) = \sup_{g\leq\delta, 0\leq x<\infty} |f(x) - 2f(x+g) + f(x+2g)|e^{-A} .$$

For the requirement in this article, we define first-order modulus of continuity as:

$$\omega_1(f, \delta, A) = \sup_{g\leq\delta, 0\leq x<\infty} |f(x) - f(x+g)|e^{-Ax}. \square$$

Theorem 5. *Let G be a subinterval having polynomials of the interval $C[0, \infty)$, and let $Q_n: G \to C[0, \infty)$ be the sequence of positive linear operators which preserves the linear functions. Let for any constant $A > 0$ and fixed $x \in [0, \infty)$, the operators Q_n satisfy.*

$$Q_n((t-x)^2e^{At}; x) \leq C(A, x) \cdot M_{n,2}^{Q_n}(x).$$

If in addition $f \in C^2[0, \infty) \cap G$ and $f'' \in$ Lip (α, A),$0 < \alpha \leq 1$, then for $x \in [0, \infty)$, we obtain:

$$\left|Q_n(f; x) - f(x) - \frac{1}{2}f''(x)M_{n,2}^{Q_n}(x)\right|$$

$$\left[e^{Ax} + \frac{C(A, x)}{2} + \frac{\sqrt{C(2A, x)}}{2}\right] \cdot M_{n,2}^{Q_n}(x)\omega_1\left(f'', \sqrt{\frac{M_{n,4}^{Q_n}(x)}{M_{n,2}^{Q_n}(x)}}, A\right). \tag{20}$$

In Theorem 5, we assume that the sequence of positive linear operators preserves linear functions. From Equation (17), we observe that S_n^A preserved only constants. By the application of Theorem 5, this is not essential. Hence, it is to show that:

$$S_n^A((t-x)^2e^{At}; x) \leq C(A, x)M_{n,2}^{S_n^A}(x) \tag{21}$$

By Lemma 4, and using Equaiton (18), for $> 2A$, we get:

$$S_n^A((t-x)^2e^{At}; x) = \frac{x^2A^2 + 2nx}{(n-A)^2}e^{Ax} \leq 8e^{Ax}M_{n,2}^{S_n^A}(x) \tag{22}$$

In view of Theorem 5, we consider the following Theorem 6 for the Phillips operators, which preserve e^{Ax}:

Theorem 6. *If $f \in G := \{f \in C[0, \infty); \| f \|_A < \infty, f \in C^2[0, \infty) \cap G$ and $f'' \in$ Lip (α, A),$0 < \alpha \leq 1$ then for $n > 2A, x \in [0, \infty)$, we mention:*

$$\left|S_n^A(f;x) - f(x) + \frac{Ax}{n}f'(x) - \left(\frac{x^2A^2}{n^2} - \frac{2xA}{n^2} + \frac{2x}{n}\right)\frac{1}{2}f''(x)\right|$$

$$\leq \left[e^{Ax} + \frac{C(A,x)}{2} + \frac{\sqrt{C(2A,x)}}{2}\right] \cdot M_{n,2}^{S_n^A}(x) \cdot \omega_1\left(f'', \sqrt{\frac{M_{n,4}^{S_n^A}(x)}{M_{n,2}^{S_n^A}(x)}}(x), A\right). \quad (23)$$

Corollary 2. Let $f, f'' \in G$, A is positive. Then for $x \in [0,\infty)$, we show:
$\lim_{n\to\infty} n[S_n^A(f;\ x) - f(x)] = x[-Af'(x) + f''(x)]$.

Remark 3. *Comparing Corollary 1 and Corollary 2, we observe that: for $A = -1$, both the results coincide.*

4. Conclusion

In the present article, the authors have obtained modified Phillips operators that preserve e^{-x} and e^{Ax}. They have presented a better approximation of modified operators than the standard Phillips operators. Further improvement in approximation can be explored by preserving other exponential functions like e^{at} and e^{bt} (a, b are real). Researchers can also find such results on other positive linear operators.

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