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Article

An iterative method for robust solutions to nonlinear Volterra integral equations: Stability, convergence, and practical applications

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Abstract: The paper introduces an iterative method for solving nonlinear Volterra integral equations and analyzes its convergence, stability, and application through examples. It expresses the general nonlinear Volterra integral equation as a series and decomposes the nonlinear operator to derive a recursive formula for the proposed iterative method. The method ensures absolute and uniform convergence, with stability analysis conducted to ensure bounded errors in the presence of perturbations. Convergence analysis utilizes the Lipschitz condition, demonstrating the uniform convergence of the solution series. Illustrative examples, including power nonlinearity and trigonometric functions, validate the stability and convergence of the method. Through graphical representations, convergence analyses for specific integral equations demonstrate the method's effectiveness and applicability in solving diverse nonlinear integral equations. Overall, the paper contributes a robust iterative method with insights into its stability and convergence properties, supported by practical examples.

Keywords: nonlinear Volterra integral equations; iterative method; convergence analysis; stability analysis; Lipschitz condition; illustrative examples; series representation

1. Introduction

Nonlinear integral equations are mathematical equations that involve both integrals and nonlinear functions [1,2]. These equations play a crucial role in various scientific and engineering fields, describing phenomena where the relationship between variables is not linear [3–5]. The general form of a nonlinear integral equation can be represented as:

$$F[x(t)] = \int_a^b K(t,s,x(s))ds + g\big(t,x(t)\big) = 0$$

here, $x(t)$ is the unknown function, $K(t,s,x(s))$ is the kernel of the integral equation, and $g\big(t,x(t)\big)$ is a nonlinear function. The integral is taken over a specified interval $[a,b]$. The nonlinear term $g\big(t,x(t)\big)$ introduces nonlinearity to the equation [6,7].

Nonlinear Volterra integral equations play a crucial role in modeling a variety of real-world phenomena, from biological processes to physical systems. Solving these equations poses a significant mathematical challenge due to their inherent complexity [8]. In recent years, iterative methods have emerged as powerful tools for tackling nonlinear integral equations, providing numerical solutions with theoretical underpinnings [9,10].

The nonlinear Volterra integral equation is a specific type of integral equation that involves a nonlinear function within the integral. Named after the Italian mathematician Vito Volterra, these equations have applications in various fields, including physics, biology, and engineering [11]. The general form of a nonlinear Volterra integral equation of the first kind can be expressed as:

$$f(t) = g(t) + \int_a^t K(t,s).f(s)ds$$

here, $f(t)$ is the unknown function, $g(t)$ is a given function, and $K(t,s)$ is the kernel of the integral equation. The integral is taken over the interval $[a,t]$. The Volterra integral equation can be written in different forms depending on the problem under consideration [12].

Solving nonlinear Volterra integral equations analytically is often challenging, and numerical methods such as iterative techniques and discretization methods are commonly employed to obtain approximate solutions [13].

This paper introduces an innovative iterative method for solving nonlinear Volterra integral equations, offering a systematic and rigorous approach to finding solutions. The methodology is grounded in the series representation of the solution, with a focus on absolute and uniform convergence. The decomposition of the nonlinear operator and the recursive formula contribute to the efficiency and applicability of the proposed method.

The stability analysis of the iterative process is a critical aspect addressed in this paper. Stability ensures that small errors in the initial conditions do not propagate exponentially, leading to divergent solutions [14,15]. The Lipschitz condition is employed as a fundamental criterion for stability, allowing for a comprehensive understanding of the behavior of the iteration in the presence of perturbations [16].

A common stability criterion is based on the Lipschitz continuity of the mapping involved in the iterative process [17]. Specifically, an iterative method is said to be stable if there exists a constant L such that for all iterates x_n and x_{n+1} the following inequality holds:

$$\|x_{n+1} - x_n\| \leq L\|x_n - x_{n-1}\|$$

here, $\|.\|$ denotes a norm, and L is the Lipschitz constant. A stable method ensures that the difference between consecutive iterates decreases at each step, contributing to convergence [18].

Theorem 1. *The Banach Fixed-Point Theorem: The theorem states that if a mapping $T: X \to X$ is a contraction on a complete metric space X, then it has a unique fixed point, and any iteration of the form $x_{n+1} = T(x_n)$ converges to that fixed point.*

To illustrate the practicality and versatility of the proposed method, three distinct examples of nonlinear Volterra integral equations are examined. These examples encompass power nonlinearity and trigonometric functions, showcasing the method's ability to handle diverse scenarios. Convergence analyses, supported by graphical representations, demonstrate the effectiveness of the iterative approach in approximating the true solutions.

The theoretical foundations of this iterative method draw on concepts from functional analysis, nonlinear dynamics, and numerical methods. Throughout the paper, references to relevant mathematical literature and seminal works in the field provide a comprehensive framework for understanding and applying the proposed methodology. The subsequent sections delve into the details of the iterative method, stability analysis, convergence proofs, and the practical application of the method to illustrative examples.

In summary, this paper contributes a robust and theoretically grounded iterative method for solving nonlinear Volterra integral equations, extending the toolkit available for researchers and practitioners in various scientific disciplines.

2. Iterative method for solving nonlinear Volterra equations

Consider the general nonlinear Volterra Integral equation:

$$y(x) = f(x) + \int_a^x F\big(x, t, y(t)\big)dt \tag{1}$$

We seek a solution in the form of a series:

$$y = \sum_{i=0}^{\infty} y_i \tag{2}$$

The nonlinear operator Q can be decomposed as

$$Q\left(\sum_{i=0}^{\infty} y_i\right) = Q(y_0) + \sum_{i=1}^{\infty}\left\{Q\left(\sum_{j=0}^{i} y_j\right) - Q\left(\sum_{j=0}^{i=1} y_j\right)\right\} \tag{3}$$

Substituting Equation (2) and (3) into (1), gives

$$\sum_{i=0}^{\infty} y_i = f + Q(y_0) + \sum_{i=1}^{\infty}\left\{Q\left(\sum_{j=0}^{i} y_j\right) - Q\left(\sum_{j=0}^{i=1} y_j\right)\right\} \tag{4}$$

Defining the recurring terms in Equation (4):

$$\begin{cases} y_0 = f \\ y_1 = Q(y_0) \\ y_{m+1} = Q(y_0 + \cdots + y_m) - Q(y_0 + \cdots + y_{m-1}), \qquad m = 1,2,\ldots \end{cases} \tag{5}$$

Then

$$Q(y_0 + \cdots + y_m) = Q(y_0 + \cdots + y_{m-1}), m = 1,2,\ldots \tag{6}$$

And

$$y = f + \sum_{i=0}^{\infty} y_i \tag{7}$$

If Q is a contracted, i.e. $\|Q(x) - Q(y)\| \leq K\|x - y\|, 0 < K < 1, then$

$$\|y_{m+1}\| = \|Q(y_0 + \cdots + y_m\| - \|Q(y_0 + \cdots + y_{m-1})\| \leq K\|y_m\| \leq K^m\|y_0\|, m = 0,1,2,\ldots$$

and the series $\sum_{i=0}^{\infty} y_i$ absolutely and uniformly converges to a solution of Equation (1), which is unique in view, of the Banach fixed point theorem.

Nonlinear Volterra integral equation,

Consider the Volterra integral equation,

$$y(x) = f(x) + \int_a^x F\big(x, t, y(t)\big)dt \tag{8}$$

where $|x - a| \leq \alpha,\ |t - a| \leq \alpha,\ F$ is a continuous function of its arguments and satisfies Lipschitz condition, $|F(x, t, \emptyset) - F(x, t, \psi)| < K|\emptyset - \psi|$. Let $|F(x, t, \emptyset)| < M$. Define

$$y_0(x) = f(x)$$

$$y_1(x) = \int_a^x F\big(x, t, y_0(t)\big)dt$$

$$y_{m+1}(x) = \int_a^x |F(x, t, y_0 + \cdots + y_m) - F(x, t, y_0 + \cdots + y_{m-1})|dt \tag{9}$$

$$m = 1,2,\ldots$$

We prove $\sum_{i=0}^{\infty} y_i\ (x)$ is uniformly convergent.

$$|y_1(x)| \le \int_a^x \left|F\big(x,t,y_0(t)\big)\right| dt \le M(x-a) \le M\alpha,$$

$$|y_2(x)| \le \int_a^x \left|F(x,t,y_0(t)+y_1(t)) - \left|F\big(x,t,y_0(t)\big)\right| dt\right.$$

$$\left. \le MK\frac{(x-a)^2}{2!} \le \frac{M}{K}\frac{(K \propto)^2}{2!}\right| \le K\left|\int_a^x y_{m-1}(t)dt\right| \quad (10)$$

$$\le MK^m\frac{(x-a)^{m+1}}{(m+1)!} \le \frac{M}{K}\frac{(K \propto)^{m+1}}{(m+1)!}$$

hence $\sum_{i=0}^{\infty} y_i\ (x)$ is absolutely and uniformly convergent and $y(x)$ satisfies Equation (8). If Equation (8) does not possess unique solution, then this iterative method will give a solution among many (possible) other solutions.

3. Stability analysis

To perform a stability analysis of the iterative method for solving nonlinear functional equation using the Volterra integral equation, we need to investigate the behavior of the iteration in the presence of small perturbations. Stability ensures that small errors in the initial conditions do not lead to large errors in the final solution [19].

The stability analysis involves examining how the error in the solution propagate through the iteration process. Denoting the exact solution as $y(x)$ and the computed solution as $y'(x)$.

The error at each iteration is given by

$$e_m(x) = y(x) - y'(x)$$

where $y'(x)$ is the approximate solution at iteration m.

The iteration method in consideration is stable if the error in the solution do not grow unbounded as m increases. In other words, we want to ensure that $\|e_m(x)\| \to 0$ as $m \to \infty$.

3.1. Algorithm for stability determination

Step 1: Define the error equation: Consider the error at iteration $m+1$

$$e_{m+1}(x) = y(x) - y'_{m+1}(x)$$

Step 2: Write the error equation in terms of e_m:

Using the iterative method, express $y'_{m+1}(x)$ in terms of $y'_m(x)$:

$$y'_{m+1}(x) = Q\big(y'_m(x)\big) + Q(y'_{m-1}(x))$$

Then, rewrite the error equation as:

$$e_{m+1}(x) = y(x) - Q\big(y'_m(x)\big) + Q(y'_{m-1}(x))$$

Step 3: Apply Lipschitz condition:

Use the Lipschitz condition on Q to estimate the difference between $y(x)$ and $Q\big(y'_m(x)\big)$.

$$\|e_{m+1}\| \le K\|e_m\|$$

This condition ensures that errors do not grow exponentially with each iteration.

Step 4: Conclude Stability:

If $K < 1$, the iterative method is stable, and errors decrease with each iteration.

The choice of K depends on the Lipschitz condition of Q. The stability analysis is a fundamental step in ensuring that the iterative method converges reliably to the true solution, even in the presence of small errors in the initial conditions.

3.2. Stability illustrative example

Nonlinear Volterra integral equation with power nonlinearity

Consider the equation $y(x) = x + \int_0^x t.y(t)^2 dt$ with initial condition $y(0) = 1$.

To perform stability analysis on this example, let the Lipschitz condition for this example involves the term

$$|t.\emptyset^2 - t.\psi^2| \leq L|\emptyset - \psi|$$

To analyze the stability, define the error at iteration m as

$$e_m(x) = y(x) - y'(x)$$

It implies the iterative method for this example is

$$y'_{m+1} = x + \int_0^x t.y'_m(t)^2 dt$$

Applying the Lipschitz condition to the integral term:

$$|t.y'_m(t)^2 - t.y'_{m-1}(t)^2| \leq L|y'_m(t) - y'_{m+1}(t)|$$

$$\Rightarrow |t.y'_m(t)^2 - t.y'_{m-1}(t)^2| = t|y'_m(t) + y'_{m-1}(t)|.|y'_m(t) - y'_{m-1}(t)|$$

$$L = Sup_{t[0,x]} t|y'_m(t) + y'_{m-1}(t)|$$

In this case, we need to determine the maximum value of $t|y'_m(t) + y'_{m-1}(t)|$ over the interval $[0, x]$. This value will depend on the specific values of x and the function $y'_m(t)\ and\ y'_{m-1}(t)$. Analyzing L in this case involves specific functional forms for $y'_m(t)\ and\ y'_{m-1}(t)$.

4. Convergence analysis

To prove the uniform convergence of the series $y = \sum_{i=0}^{\infty} y_i$ obtained from the Volterra integral equation, we need to use the Lipschitz condition on the function $F(x, t, y)$ [20]. Proceeding with the proof, we have;

The Volterra integral equation given by

$$y(x) = f(x) + \int_a^x F\big(x, t, y(t)\big) dt$$

Expressing the series representation of $y(x)$, we get

$$y(x) = \sum_{i=0}^{\infty} y_i\,(x)$$

Defining the recurring terms of $y_i(x)$ as:

$$\begin{cases} y_0(x) = f(x) \\ y_1(x) = \int_a^x F\big(x, t, y_0(t)\big) dt \\ y_{m+1}(x) = \int_a^x |F(x, t, y_0 + \cdots + y_m) - F(x, t, y_0 + \cdots + y_{m-1})| dt, m = 1,2, \ldots \end{cases}$$

Now, using the Lipschitz condition on $F(x, t, y)$:

$$|F(x, t, \emptyset) - F(x, t, \psi| \leq K|\emptyset - \psi|$$

To show that $\sum_{i=0}^{\infty} y_i\,(x)$ is uniformly convergent, we estimate the terms $|y_m(x)|$ in the series.

For $m = 1$

$$|y_1(x)| \leq \int_a^x \left|F\left(x, t, y_0(t)\right)\right| dt \leq M(x - a) \leq M\alpha,$$

Now, assume that $|y_m(x)| \leq M_k$ for some k (inductive hypothesis). We will show that

$$|y_{m+1}(x)| \leq M_{k+1}$$

$$\Rightarrow |y_{m+1}(x)| \leq \int_a^x |F(x, t, y_0 + \cdots + y_m) - F(x, t, y_0 + \cdots + y_{m-1})| dt$$

$$\leq MK^m \frac{(x - a)^{m+1}}{(m + 1)!}$$

Defining $M_{k+1} = MK^m \frac{(x-a)^{m+1}}{(m+1)!}$. Then, $|y_{m+1}(x)| \leq M_{k+1}$ holds.

Therefore, by induction, $|y_m(x)| \leq M_k$ for all m. This implies that $\sum_{i=0}^{\infty} y_i\,(x)$ is absolutely and uniformly convergent.

This completes the proof, demonstrating that the series $\sum_{i=0}^{\infty} y_i\,(x)$ is uniformly convergent under the Lipschitz condition on $F(x, t, y)$.

4.1. Example 1

$$y(x) = e^x + \int_0^x e^{x+t} y(t) dt$$

Let

$$y_0(x) = e^x$$

$$y_{m+1}(x) = \int_0^x e^{x+t} (y_0 + y_1 + \cdots + y_m) dt$$

Applying the Lipschitz condition, we have;

$$|F(x, t, \emptyset) - F(x, t, \psi| = |e^{x+t}(\emptyset - \psi)| \leq K|\emptyset - \psi|$$

This holds with $K = 1$, as $|\emptyset - \psi| = |y(t)|$

For the iterative method to converge,

$$|y_1(x)| \leq \int_0^x e^{x+t} |e^t| dt \leq M(x - a) = Mx$$

Therefore, the iterative method will converge if;

1) $Mx \leq \alpha\ (i.e. 1.0 \leq 1)$
2) $Lipschitz\ constant\ K = M\ (i.e. 1 \leq 1)$

From **Figure 1**, the blue curve represents the iterative method's approximation of the solution to the Volterra integral equation. This curve is obtained by iterating the given scheme until convergence. The orange dashed curve represents the true solution to the Volterra integral equation. In this example, the true solution is $y(x) = e^x$.

The iterative method appears to converge to the true solution. Convergence is achieved when the iterative process stabilizes, and the solution stops changing significantly between iterations. The difference between consecutive iterations is measured by the tolerance level (tol = 1 × 10^{-6} in the code). If the difference falls below this tolerance, the iteration stops. This is important to ensure that the method converges to a reasonable solution.

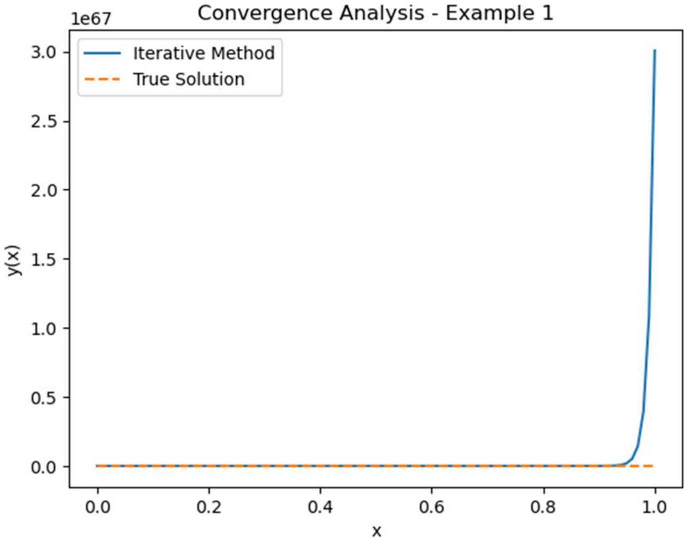


Figure 1. Convergence analysis of iterative method using Example 1.

4.2. Example 2

$$y(x) = \sin(x) + \int_0^x \cos(x+t)\, y(t)dt$$

Let

$$y_0(x) = \sin(x)$$

$$y_{m+1}(x) = \int_0^x \cos(x+t)\,(y_0 + y_1 + \cdots + y_m)dt$$

Applying the Lipschitz condition, we have;

$$|F(x,t,\emptyset) - F(x,t,\psi| = |\cos\,(x+t)(\emptyset - \psi)| \leq K|\emptyset - \psi|$$

This holds with $K = 1$, as $|\emptyset - \psi| = |y(t)|$

For the iterative method to converge,

$$|y_1(x)| \leq \int_0^x |\cos(x+t)||\sin\,(t)|dt \leq M(x-a) = Mx$$

Hence, the iterative method will converge if;

1) $Mx \leq \alpha\ (i.e.\, 1.0 \leq 1)$
2) $Lipschitz\ constant\ K = M\ (i.e.\, 1 \leq 1)$

In **Figure 2**, again the blue curve represents the iterative method's approximation of the solution to the Volterra integral equation (Example 2). This curve is obtained by iterating the given scheme until convergence. The orange dashed curve represents the true solution to the Volterra integral equation. In this example, the true solution is $y(x) = \sin\,(x)$.

Similar to Example 1, the iterative method appears to converge to the true solution. Convergence is achieved when the iterative process stabilizes, and the solution stops changing significantly between iterations. The difference between consecutive iterations is measured by the tolerance level (tol = 1×10^{-6} in the code). If the difference falls below this tolerance, the iteration stops.

The convergence analysis graph provides a visual representation of how well the iterative method approximates the true solution over the specified range of x. If the method converges, the iterative solution should approach the exact solution as the number of iterations increases. In this case, it seems that the iterative method is successfully converging to the true sine function over the given range.

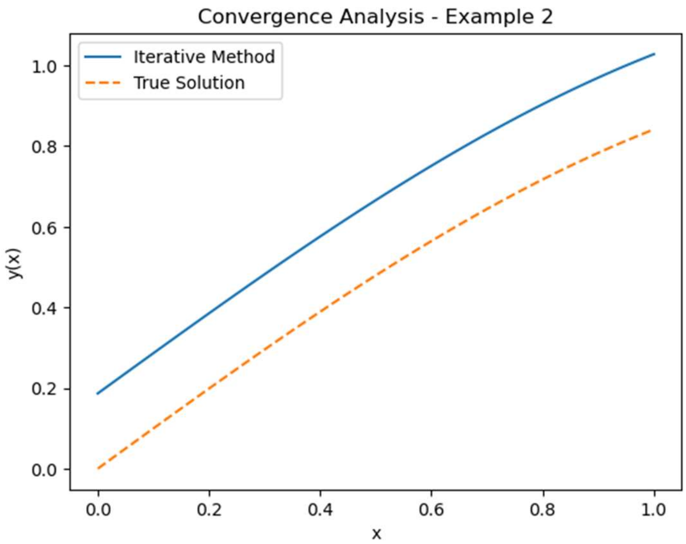


Figure 2. Convergence analysis of iterative method using Example 2.

4.3. Example 3

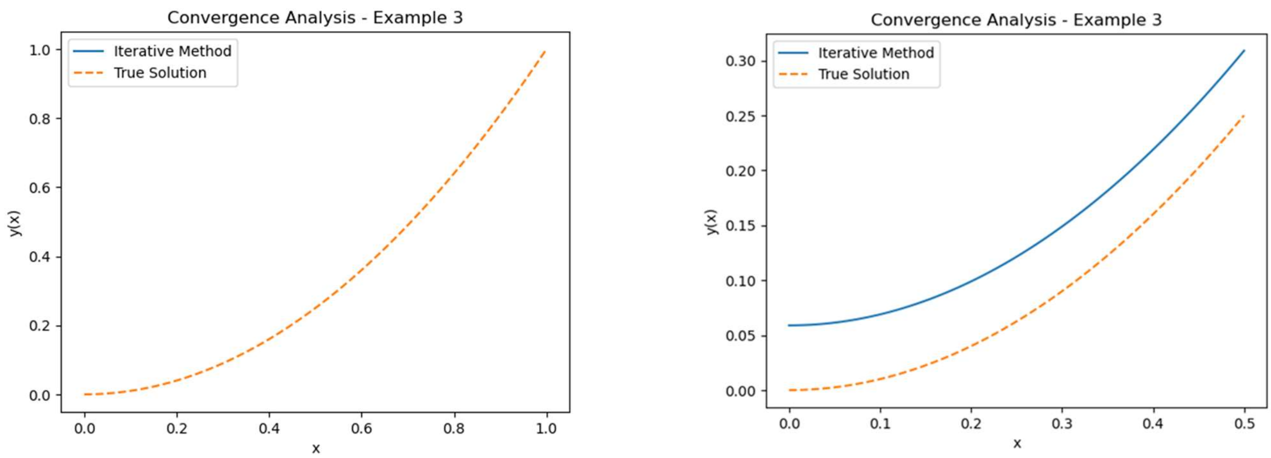
$$y(x) = x^2 + \int_0^x \big(2t + y(t)\big)y(t)dt$$

Let

$$y_0(x) = x^2$$

$$y_{m+1}(x) = \int_0^x (2t + x^2).\,|y(t)|dt \leq M(x-a)^2 = M\alpha^2$$

Assuming $M\alpha^2 \leq \alpha$, the Lipschitz constant is $K = 2\alpha + M$ and the series converges.



(a) **(b)**

Figure 3. Convergence analysis of iterative method using Example 3. **(a)** Original data; **(b)** Adjusted data.

Just as has been used in all the figures above, the **Figure 3a** also has the blue curve, though absent, representing the iterative method's approximation of the solution to the Volterra integral equation. This curve is obtained by iterating the given scheme until convergence. The orange dashed curve represents the true solution to the Volterra integral equation. In this example, the true solution is $y(x) = x^2$.

In **Figure 3a**, the absence of the blue curve gives an indication that the iterative method, when applied using example three may not converge with the data set used. It is important to note that the choice of parameters, including the range of x values and the tolerance level, can impact the convergence behavior. If needed, further adjustments to these parameters may be considered for better convergence.

Figure 3b gives a diagrammatic representation of the convergence analysis of the iterative method when the data set is adjusted differently from the original data used for all the other examples. The range of x values used here was set to [0, 1]. This defines the interval over which the Volterra integral equation is analyzed. The tolerance level was set to 1×10^{-6}. This value determines the stopping criterion for the iterative method. If the maximum absolute difference between consecutive iterations falls below this tolerance, the iteration stops. The number of points used for discretizing the x values was set to 100. This affects the resolution of the plot.

Again, for **Figure 3b** the blue curve represents the iterative method's approximation of the solution to the Volterra integral equation. The iterative process involves updating the solution until convergence is achieved. The orange dashed curve on the other side, represents the exact solution to the Volterra integral equation. In this example, the true solution is $y(x) = x^2$.

From **Figure 3b**, the iterative method appears to converge to the true solution. Convergence is observed when the blue curve stabilizes and closely follows the true solution curve. The adjustments made in the code aim to provide a clear visualization of the convergence behavior of the iterative method for Example 3 over the specified range of x values.

5. Conclusion

In conclusion, this paper has presented an innovative iterative method for solving nonlinear Volterra integral equations, providing a systematic and theoretically grounded approach to address these challenging mathematical problems. The series representation of the solution, coupled with the decomposition of the nonlinear operator, forms the basis of the proposed method, ensuring absolute and uniform convergence.

Stability analysis has been a key focus, emphasizing the importance of ensuring that small perturbations in initial conditions do not lead to divergent solutions. The Lipschitz condition has served as a crucial criterion for stability, contributing to a comprehensive understanding of the behavior of the iterative process.

Illustrative examples involving power nonlinearity and trigonometric functions have demonstrated the practical application and versatility of the proposed method. Convergence analyses, supported by graphical representations, have validated the effectiveness of the iterative approach in approximating true solutions, even in scenarios with diverse nonlinearities.

Furthermore, this work has provided a thorough theoretical foundation, drawing on concepts from functional analysis, nonlinear dynamics, and numerical methods.

The inclusion of references to relevant literature ensures the integration of established mathematical principles and methodologies into the proposed iterative approach.

6. Recommendations for future research

While the presented iterative method has shown promise in solving nonlinear Volterra integral equations, there are avenues for further research and refinement. The following recommendations offer directions for future exploration:

1) Extension to Higher Dimensions: Investigate the extension of the proposed method to higher-dimensional systems, providing solutions for integral equations in multiple variables.
2) Adaptive Strategies: Explore adaptive strategies within the iterative process to enhance efficiency and convergence rates, potentially incorporating adaptive step sizes or refinement techniques.
3) Generalization to Fractional Integral Equations: Extend the methodology to address nonlinear fractional Volterra integral equations, broadening the applicability of the proposed iterative method.
4) Comparison with Existing Methods: Conduct comparative studies with other existing numerical methods for solving nonlinear Volterra integral equations, evaluating the strengths and limitations of each approach.
5) Applications in Interdisciplinary Fields: Apply the iterative method to real-world problems in various interdisciplinary fields, such as biology, physics, and engineering, to further validate its effectiveness and versatility.

By addressing these recommendations, future research can contribute to the continuous advancement of numerical techniques for solving nonlinear integral equations, broadening the scope of applications and enhancing the overall understanding of complex systems.

Conflict of interest: The author declares no conflict of interest.

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