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Article

On galaxies of sequences of toeplitz matrix solutions of the Diophantine Equation $X^n + Y^n = Z^n, n \geq 3$

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Abstract: We construct the galaxies of sequences of Toeplitz matrix solutions of the Diophantine equation $X^n + Y^n = Z^n, n \geq 3$, linked to Pythagorean triples.

Keywords: matrices of integers; solutions of Diophantine equations; Fermat's equations
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1. Introduction and main result

It is well known that there are many solutions in integers to the equation $x^2 + y^2 = z^2$, for instance (3,4,5); (5,12,13). Around 1500 B.C, the Babylonians were aware of the solution (4961, 6480, 8161) and the Egyptians knew the solutions (148, 2736, 2740) and (514, 66048, 66050). Also Greek mathematicians were attracted to the solutions of this equation. In 2021, Mouanda introduced a powerful new method of generating galaxies of sequences of Pythagorean triples [1]. Fermat's Last Theorem states that the equation

$$x^n + y^n = z^n, n \geq 3,$$

has no positive integer solutions [2–4]. In 1966, Domiaty proved that the equation $X^4 + Y^4 = Z^4$ is solvable in $M_2(\mathbb{Z})$ [5]. Let $GL_n(\mathbb{Z})$ be the group of units of ring $M_n(\mathbb{Z})$. Denote by

$$SL_n(\mathbb{Z}) = \{A \in M_n(\mathbb{Z}) : detA = 1\}.$$

In 1989, Vaserstein investigated the question of the solvability of the Diophantine equation

$$X^n + Y^n = Z^n, n \geq 2, \tag{1}$$

for matrices of the group $GL_2(\mathbb{Z})$ [6]. In 1993, Frejman studied the solvability of the Diophantine Equation (1) in the set of positive integer powers of a matrix A with elements $a_{11} = 0, a_{12} = a_{21} = a_{22} = 1$ [7]. In 1995, the same case was studied by Grytczuk [8]. The same year, Khazanov proved that in $GL_3(\mathbb{Z})$ solutions of the Diophantine Equation (1) do not exist if n is a multiple of either 21 or 96, and in $SL_3(\mathbb{Z})$ solutions do not exist if n is a multiple of 48 [9]. In 1996, Qin gave another proof of Khazanov's result on the solvability of the Diophantine Equation (1) in $SL_2(\mathbb{Z})$ [10]. In 2002, Patay and Szakacs described the periodic elements in $GL_2(\mathbb{Z})$ and gave the answer to some problems concerning the Diophantine Equation (1) in matrix groups and in irreducible elements of matrix rings [11]. In 2021, Mao-Ting and Jie proved that Fermat's matrix equation has many solutions in a set of 2-by-2 positive semi-definite integral matrices, and has no nontrivial solutions in some classes including 2-by-2 symmetric rational and stochastic quadratic field matrices [12]. Fermat's Last Theorem has been extended to the field of complex polynomials of one variable [13]. In 2022, Mouanda, Kangni and Tsiba proved the equation $X^2 + Y^2 = Z^2$ admits matrix solutions in the set of circulant matrices with positive integers as entries [14]. The same year, Mouanda noticed that

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^9 + \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^9 = A^9$$

with

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^9.$$

Pythagorean triples have many applications in Cryptography.

In this paper, we introduce a new method of generating the galaxies of sequences of Toeplitz matrix solutions of the Diophantine equation

$$X^n + Y^n = Z^n, n \geq 3, X, Y, Z \in M_n(\mathbb{N}),$$

linked to Pythagorean triples.

Theorem 1.1. *There exists an infinite number of galaxies of sequences $(X_k, Y_k, Z_k)_{k \geq 0}$ of Toeplitz matrix triples of $M_n(\mathbb{N})$ such that*

$$X_k^n + Y_k^n = Z_k^n, X_k, Y_k, Z_k \in M_n(\mathbb{N}), k, n \in \mathbb{N}, n \geq 3.$$

2. Proof of the main result

In this section, we investigate the Toeplitz matrix solutions of the equation $X^n + Y^n = Z^n, n \geq 3$.

Definition 2.1. *A triple $(x, y, z) \in \mathbb{N}^3$ is said Pythagorean if $x^2 + y^2 = z^2$.*

Denote by $\mathcal{F}_{2,2,2}(\mathbb{N}) = \left\{(a, b, c) \in \mathbb{N}^3 : a^2 + b^2 = c^2\right\}$ is the set of Pythagorean triples. Therefore, we can denote by

$$\mathcal{F}_{n,n,n}(M_m(\mathbb{C})) = \left\{(A,B,C) \in M_m(\mathbb{C})^3 : A^n + B^n = C^n\right\}, n \geq 2, n \in \mathbb{N}.$$

In 2022, Mouanda proved that the universe

$$\mathcal{F}_{2,2,2}(M_n(\mathbb{N})) = \left\{(X,Y,Z) : X^2 + Y^2 = Z^2, X, Y, Z \in M_n(\mathbb{N})\right\}$$

has an infinite number of elements [14]. Fermat's Last Theorem allows us to claim that the Diophantine equation $x^n + y^n = z^n, n \geq 3$, has no positive integer solutions. The idea of finding all the sequences $(X_k, Y_k, Z_k)_{k\geq 0}$ of matrix triples with positive integers as entries which satisfy

$$X_k^n + Y_k^n = Z_k^n, k, n \in \mathbb{N}, n \geq 3,$$

is an unthinkable idea. Mouanda's recent work on this direction allows us to believe that these sequences of matrix triple solutions do really exist because they are linked to Pythagorean triples.

Definition 2.2. *A finite matrix $A = [a_{i,j}]_{i,j=1}^n$ is called a Toeplitz matrix if $a_{i+1,j+1} = a_{i,j}$.*

Each descending diagonal from left to right of a Toeplitz matrix is constant. For instance, the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

is a Toeplitz matrix. Let $A_{\alpha,n} = [a_{i,j}]_{i,j=1}^n \in M_n(\mathbb{N})$ be a Toeplitz matrix such that

$$\begin{cases} a_{1,3} = 1, \\ a_{n-1,1} = \alpha, \\ a_{i,j} = 0, \forall i \notin \{1, n-1\}, \forall j \notin \{1, 3\}. \end{cases}$$

In other words,

$$A_{\alpha,n} = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix} \in M_n(\mathbb{N}), \alpha \in \mathbb{N}.$$

The matrix $A_{\alpha,n}$ is called a Rare matrix of order n and index 2 [15]. The matrix $A_{\alpha,n}$ is an $n \times n-$ matrix. For example,

$$A_{\alpha,4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix}, A_{\alpha,5} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \end{pmatrix},$$

$$A_{\alpha,6} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \end{pmatrix}$$

are Rare matrices of index 2. The matrices of the set $\{A_{\alpha,n} : \alpha \in \mathbb{N}\}$ allow us to construct the matrix triple solutions of the Diophantine equation $X^n + Y^n = Z^n$. The most interesting part of this study is the seize of the matrix solutions. In our case, n is the seize of the matrix solutions. This observation allows us to claim that it is practically impossible to write down on a computer or on a board the matrix solutions of the Diophantine equation $X^n + Y^n = Z^n$ for n sufficiently large.

Remark 2.3. *Let*

$$A_{\alpha,n} = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix} \in M_n(\mathbb{N}), \alpha \in \mathbb{N},$$

be a Rare matrix of order n *and index 2. Then*

$$A_{\alpha,n}^n = \begin{pmatrix} \alpha^2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \alpha^2 \end{pmatrix} \in M_n(\mathbb{N}).$$

- Assume that $(a, b, c) \in \mathcal{F}_{2,2,2}(\mathbb{N})$. In other words, $a^2 + b^2 = c^2$. Let us consider the Toeplitz matrices

$$A_a = \begin{pmatrix} 0 & 0 & 1 \\ a & 0 & 0 \\ 0 & a & 0 \end{pmatrix}, A_b = \begin{pmatrix} 0 & 0 & 1 \\ b & 0 & 0 \\ 0 & b & 0 \end{pmatrix}, A_c = \begin{pmatrix} 0 & 0 & 1 \\ c & 0 & 0 \\ 0 & c & 0 \end{pmatrix}.$$

A simple calculation shows that

$$A_a^3 = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{pmatrix}, A_b^3 = \begin{pmatrix} b^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & b^2 \end{pmatrix}, A_c^3 = \begin{pmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}.$$

It is clear that

$$A_a^3 + A_b^3 = \begin{pmatrix} a^2+b^2 & 0 & 0 \\ 0 & a^2+b^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} = \begin{pmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix} = A_c^3.$$

Therefore, $A_a^3 + A_b^3 = A_c^3$. This implies that $(A_a, A_b, A_c) \in \mathcal{F}_{3,3,3}(M_3(\mathbb{N}))$. Finally, every Pythagorean triple generates a Toeplitz matrix solution of the Diophantine equation $X^3 + Y^3 = Z^3$.

- Assume that $(a, b, c) \in \mathcal{F}_{2,2,2}(\mathbb{N})$. In other words, $a^2 + b^2 = c^2$. Let us consider the Toeplitz matrices

$$A_a = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, A_b = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}, A_c = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}.$$

A simple calculation shows that

$$A_a^4 = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}, A_b^4 = \begin{pmatrix} b^2 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 \\ 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & b^2 \end{pmatrix},$$

$$A_c^4 = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & c^2 & 0 & 0 \\ 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & c^2 \end{pmatrix}.$$

It is clear that

$$A_a^4 + A_b^4 = \begin{pmatrix} a^2+b^2 & 0 & 0 & 0 \\ 0 & a^2+b^2 & 0 & 0 \\ 0 & 0 & a^2+b^2 & 0 \\ 0 & 0 & 0 & a^2+b^2 \end{pmatrix} = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & c^2 & 0 & 0 \\ 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & c^2 \end{pmatrix} = A_c^4.$$

Therefore, $A_a^4 + A_b^4 = A_c^4$. This implies that $(A_a, A_b, A_c) \in \mathcal{F}_{4,4,4}(M_4(\mathbb{N}))$. Finally, every Pythagorean triple generates a Toeplitz matrix solution of the equation $X^4 + Y^4 = Z^4$.

- Assume that $(a, b, c) \in \mathcal{F}_{2,2,2}(\mathbb{N})$. In other words, $a^2 + b^2 = c^2$. Let us consider the Toeplitz matrices

$$A_a = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \end{pmatrix}, A_b = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \end{pmatrix},$$

$$A_c = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 \end{pmatrix}.$$

A simple calculation shows that

$$A_a^5 = \begin{pmatrix} a^2 & 0 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 & 0 \\ 0 & 0 & a^2 & 0 & 0 \\ 0 & 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & 0 & a^2 \end{pmatrix}, A_b^5 = \begin{pmatrix} b^2 & 0 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 & 0 \\ 0 & 0 & b^2 & 0 & 0 \\ 0 & 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & 0 & b^2 \end{pmatrix}$$

and

$$A_c^5 = \begin{pmatrix} c^2 & 0 & 0 & 0 & 0 \\ 0 & c^2 & 0 & 0 & 0 \\ 0 & 0 & c^2 & 0 & 0 \\ 0 & 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & 0 & c^2 \end{pmatrix}.$$

Therefore, $A_a^5 + A_b^5 = A_c^5$. This implies that $(A_a, A_b, A_c) \in \mathcal{F}_{5,5,5}(M_5(\mathbb{N}))$. Finally, every Pythagorean triple generates a Toeplitz matrix solution of the equation $X^5 + Y^5 = Z^5$.

- Assume that $(a, b, c) \in \mathbb{F}_2(\mathbb{N})$. In other words, $a^2 + b^2 = c^2$. Let us consider the Toeplitz matrices

$$A_a = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_b = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$A_c = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

A simple calculation shows that

$$A_a^8 = \begin{pmatrix} a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^2 \end{pmatrix},$$

$$A_b^8 = \begin{pmatrix} b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^2 \end{pmatrix}$$

and

$$A_c^8 = \begin{pmatrix} c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c^2 \end{pmatrix}.$$

It is quiet clear that $A_a^8 + A_b^8 = A_c^8$. In particular, the matrices

$$A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$A_5 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

satisfy

$$A_3^8 = \begin{pmatrix} 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix},$$

$$A_4^8 = \begin{pmatrix} 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 \end{pmatrix}$$

and

$$A_5^8 = \begin{pmatrix} 25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 \end{pmatrix}.$$

It follows that $A_3^8 + A_4^8 = A_5^8$. Let us consider the galaxies of sequences of Pythagorean triples [1]

$$Za(\alpha, \mathbb{N}) = \begin{bmatrix} x_k(\alpha, a) = \alpha^2 + 6\alpha a^k \\ y_k(\alpha, a) = 6\alpha a^k + 18a^{2k} \\ z_k(\alpha, a) = \alpha^2 + 6\alpha a^k + 18a^{2k} \\ k \in \mathbb{N}, a \in \mathbb{N} \end{bmatrix}, \alpha \in \mathbb{N}.$$

It is clear that $x_k(\alpha, a)^2 + y_k(\alpha, a)^2 = z_k(\alpha, a)^2, a, \alpha \in \mathbb{N}$. The matrices

$$A_{x_k(\alpha,a)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ x_k(\alpha, a) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_k(\alpha, a) & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{y_k(\alpha,a)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ y_k(\alpha, a) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_k(\alpha, a) & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$A_{z_k(\alpha,a)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ z_k(\alpha,a) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_k(\alpha,a) & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

satisfy $A^8_{x_k(\alpha,a)} + A^8_{y_k(\alpha,a)} = A^8_{z_k(\alpha,a)}$. We can claim that

$$La(\alpha, M_8(\mathbb{N})) = \begin{bmatrix} A_{x_k(\alpha,a)} \\ A_{y_k(\alpha,a)} \\ A_{z_k(\alpha,a)} \\ k \in \mathbb{N}, a \in \mathbb{N} \end{bmatrix} \subset \mathbb{F}_8(M_8(\mathbb{N})), \alpha \in \mathbb{N},$$

are galaxies of matrix solutions of the Diophantine equation $X^8 + Y^8 = Z^8$. This implies that $(A_{x_k(\alpha,a)}, A_{y_k(\alpha,a)}, A_{z_k(\alpha,a)}) \in \mathcal{F}_{8,8,8}(M_8(\mathbb{N})), k \in \mathbb{N}$. We can now prove our main result.

Proof of Theorem 1.1

Let

$$A_{\alpha,n} = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix} \in M_n(\mathbb{N}), \alpha \in \mathbb{N},$$

be a Rare matrix of order n and index 2. Remark 2.3. allows us to claim that

$$A^n_{\alpha,n} = \begin{pmatrix} \alpha^2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \alpha^2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \alpha^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \alpha^2 \end{pmatrix} \in M_n(\mathbb{N}).$$

The structure of the matrix $A_{\alpha,n}$ could allow us to construct the Toeplitz matrix solutions of the Diophantine equation $X^n + Y^n = Z^n$. Indeed, let us consider the galaxies of sequences of Pythagorean triples [1]

$$Za(\alpha, \mathbb{N}) = \left[\begin{array}{c} x_k(\alpha, a) = \alpha^2 + 6\alpha a^k \\ y_k(\alpha, a) = 6\alpha a^k + 18a^{2k} \\ z_k(\alpha, a) = \alpha^2 + 6\alpha a^k + 18a^{2k} \\ k \in \mathbb{N}, a \in \mathbb{N} \end{array} \right], \alpha \in \mathbb{N}.$$

Denote by

$$A_{x_k(\alpha,a),n} = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ x_k(\alpha, a) & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & x_k(\alpha, a) & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{y_k(\alpha,a),n} = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ y_k(\alpha, a) & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & y_k(\alpha, a) & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$A_{z_k(\alpha,a),n} = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ z_k(\alpha, a) & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & z_k(\alpha, a) & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

A simple calculation shows that

$$A^n_{x_k(\alpha,a),n} = \begin{pmatrix} x_k(\alpha,a)^2 & 0 & 0 & 0 & \dots & 0 \\ 0 & x_k(\alpha,a)^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & x_k(\alpha,a)^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & x_k(\alpha,a)^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & x_k(\alpha,a)^2 \end{pmatrix},$$

$$A^n_{y_k(\alpha,a),n} = \begin{pmatrix} y_k(\alpha,a)^2 & 0 & 0 & 0 & \dots & 0 \\ 0 & y_k(\alpha,a)^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & y_k(\alpha,a)^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & y_k(\alpha,a)^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & y_k(\alpha,a)^2 \end{pmatrix}$$

and

$$A^n_{z_k(\alpha,a),n} = \begin{pmatrix} z_k(\alpha,a)^2 & 0 & 0 & 0 & \dots & 0 \\ 0 & z_k(\alpha,a)^2 & 0 & 0 & \dots & 0 \\ 0 & 0 & z_k(\alpha,a)^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & z_k(\alpha,a)^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & z_k(\alpha,a)^2 \end{pmatrix}.$$

Therefore, $A^n_{x_k(\alpha,a),n} + A^n_{y_k(\alpha,a),n} = A^n_{z_k(\alpha,a),n}, \alpha, n, a, k \in \mathbb{N}$. Finally,

$$\Gamma(\alpha, M_n(\mathbb{N})) = \begin{bmatrix} A_{x_k(\alpha,a),n} \\ A_{y_k(\alpha,a),n} \\ A_{z_k(\alpha,a),n} \\ k, a \in \mathbb{N} \end{bmatrix} \subset \mathbb{F}_n(M_n(\mathbb{N})), \alpha, n \in \mathbb{N},$$

are galaxies of Toeplitz matrix solutions of the Diophantine equation $X^n + Y^n = Z^n$. $\square$

The equation $X^n + Y^n = Z^n, n \geq 3$, always has an infinite number of matrix solutions in $M_{nm}(\mathbb{N})$. The sequences of the matrix triples $(X_k(\alpha, A)$, $Y_k(\alpha, A)$, $Z_k(\alpha, A))_{k\in\mathbb{N}}$ of the galaxy

$$Wa(\alpha, M_m(\mathbb{N})) = \begin{bmatrix} X_k(\alpha, A) = \alpha^4 I_m + 2\alpha^2 A^k \\ Y_k(\alpha, A) = 2\alpha^2 A^k + 2A^{2k} \\ Z_k(\alpha, A) = \alpha^4 I_m + 2\alpha^2 A^k + 2A^{2k} \\ k \in \mathbb{N}, A \in M_m(\mathbb{N}) \end{bmatrix}, \alpha \in \mathbb{N},$$

could allow us to construct the matrix solutions in $M_{n\times m}(\mathbb{N})$. In our case, $(A_{X_k(\alpha,A),n}$, $A_{Y_k(\alpha,A),n}$, $A_{Z_k(\alpha,A),n}) \in \mathcal{F}_{n,n,n}(M_{nm}(\mathbb{N}))$.

Our next work will be focused on finding all the structures of matrix solutions of this Diophantine equation for n sufficiently large.

Conflict of interest: The author declares no conflict of interest.

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