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ORIGINAL RESEARCH ARTICLE

Inhomogeneous spatial patterns in diffusive predator-prey system with spatial memory and predator-taxis

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ABSTRACT

In this paper, by introducing predator-taxis into the diffusive predator-prey system with spatial memory, then we study the inhomogeneous spatial patterns of this system. Since in this system, the memory delay appears in the diffusion term, and the diffusion term is nonlinear, the classical normal form of Hopf bifurcation for the reaction-diffusion system with delay can't be applied to this system. Thus, in this paper, we first derive an algorithm for calculating the normal form of Hopf bifurcation for this system. Then in order to illustrate the effectiveness of our newly developed algorithm, we consider the diffusive Holling-Tanner model with spatial memory and predator-taxis. The stability and Hopf bifurcation analysis of this model are investigated, and the direction and stability of Hopf bifurcation periodic solution are also studied by using our newly developed algorithm for calculating the normal form of Hopf bifurcation. At last, we carry out some numerical simulations to verify our theoretical analysis results, and two stable spatially inhomogeneous periodic solutions corresponding to the mode-1 and mode-2 Hopf bifurcations are found.

Keywords: predator-prey system; memory delay; predator-taxis; Hopf bifurcation; normal form; periodic solutions
MSC Classification: 35B10; 37G05; 37L10; 92D25

1. Introduction

The reaction-diffusion systems based on the Fick's law have been widely used in physics, chemistry and biology[1–3]. More precisely, based on the Fick's law, that is the movement flux is in the direction of negative gradient of the density distribution function, the predator-prey model with cross-diffusion considering two different prey behaviors' transition[4], the diffusive predator-prey model in heterogeneous environment[5,6], the diffusive predator-prey model with a protection zone[7], the diffusive predator-prey model with prey social behavior[8], the diffusive predator-prey model with protection zone and predator harvesting[9], the diffusive predator-prey model with Bazykin functional response have been studied by many researchers[10]. Furthermore, a predator-prey meta-population model is studied by Bajeux et al.[11]. In order to include the episodic-like spatial memory of animals, Shi et al.[12] directed movement toward the negative gradient of the density distribution function at the past time, and they proposed the following diffusive model with spatial memory

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$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1 u_{xx}(x,t) + d_2(u(x,t)u_x(x,t-\tau))_x + f(u(x,t)), & x \in \Omega, t > 0, \\ \dfrac{\partial u}{\partial \mathbf{n}}(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,t) = u_0(x,t), & x \in \Omega, -\tau \le t \le 0, \end{cases} \tag{1}$$

where $u(x,t)$ is the population density at spatial location x and time t, d_1 and d_2 are the Fickian diffusion coefficient and the memory-based diffusion coefficient, respectively, $\Omega \subset \mathbb{R}$ is a smooth and bounded domain, $u_0(x,t)$ is the initial function, $u_{xx}(x,t) = \partial^2 u(x,t)/\partial x^2, u_x(x,t) = \partial u(x,t)/\partial x, u_x(x,t-\tau) = \partial u(x,t-\tau)/\partial x$, $u_{xx}(x,t-\tau) = \partial^2 u(x,t-\tau)/\partial x^2$, and n is the outward unit normal vector at the smooth boundary $\partial\Omega$. Here, the time delay $\tau > 0$ represents the averaged memory period, which is usually called as the memory delay, and $f(u(x,t))$ describes the chemical reaction, and the biological birth or death. Moreover, in order to further investigate the influence of memory delay on the stability of the positive constant steady state, on the basis of model (1), Shi et al.[13] studied the spatial memory diffusion model with memory and maturation delays. Furthermore, Song et al.[14] have considered a diffusive predator-prey system with memory-based diffusion and Holling type-II functional response, and by carrying out some numerical simulations, the stable spatially inhomogeneous periodic solutions and the transition from the unstable mode-2 spatially inhomogeneous periodic solution to the stable mode-1 spatially inhomogeneous periodic solution are found.

For the general predator-prey models in ecology, apart from the random diffusion of the predator and prey populations, the spatial movement of predator and prey populations also occurs, which is usually shown as the predator pursuing prey and prey escaping from predator[15]. The pursuit and evasion between the predator and prey populations also have a strong impact on the movement pattern of the predator and prey populations[16–18]. By noticing that such movement is not random but directed, i.e., predator moves toward the gradient direction of prey distribution, which is called prey-taxis, or prey moves opposite to the gradient of predator distribution, which is called predator-taxis. Recently, the predator-prey model with prey-taxis[19–24], the predator-prey model with indirect prey-taxis[25,26], the predator-prey model with predator-taxis[27] and the predator-prey model with indirect predator-taxis[28] have been researched. Especially, Wang et al.[15] considered the following diffusive predator-prey model with both predator-taxis and prey-taxis, and their proposed model is

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d u_{xx}(x,t) + \xi(u(x,t)v_x(x,t))_x + f(u(x,t),v(x,t)), & x \in \Omega, t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = v_{xx}(x,t) - \eta(v(x,t)u_x(x,t))_x + g(u(x,t),v(x,t)), & x \in \Omega, t > 0, \\ \dfrac{\partial u(x,t)}{\partial \mathbf{n}} = \dfrac{\partial v(x,t)}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \ge 0, v(x,0) = v_0(x) \ge 0, & x \in \Omega, \end{cases} \tag{2}$$

where $\Omega = (0, \ell\pi)$ with $\ell \in \mathbb{R}^+, u(x,t)$ and $v(x,t)$ represent the densities of prey and predator at the location x and time t, respectively, $u_0(x)$ and $v_0(x)$ are the initial functions, d is the rescaled diffusion coefficient for the prey population, and the diffusion coefficient of the predator population is rescaled as 1. Furthermore, the term $\xi(u(x,t)v_x(x,t))_x$ represents the prey moves away from predator, and $\xi > 0$ is the intrinsic predator-taxis rate. The term $-\eta(v(x,t)u_x(x,t))_x$ represents the predator moves towards prey, and $\eta > 0$ is the intrinsic prey-taxis rate. Therefore, by combining with models (1) and (2), and by considering that the spatial memory and predator-taxis, we proposed the following diffusive predator-prey model with spatial memory and predator-taxis subjects to the homogeneous Neumann boundary condition

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=d_{11}u_{xx}(x,t)+\xi(u(x,t)v_x(x,t))_x+f(u(x,t),v(x,t)), & x\in(0,\ell\pi),t>0,\\ \dfrac{\partial v(x,t)}{\partial t}=d_{22}v_{xx}(x,t)-d_{21}(v(x,t)u_x(x,t-\tau))_x+g(u(x,t),v(x,t)), & x\in(0,\ell\pi),t>0,\\ u_x(0,t)=u_x(\ell\pi,t)=v_x(0,t)=v_x(\ell\pi,t)=0, & t>0,\\ u(x,t)=u_0(x,t),v(x,t)=v_0(x,t), & x\in(0,\ell\pi),-\tau\le t\le 0,\end{cases} \tag{3}$$

where $d_{11}>0$ and $d_{22}>0$ are the random diffusion coefficients, $d_{21}>0$ is the memory-based diffusion coefficient, $u_0(x,t)$ and $v_0(x,t)$ are the initial functions, and $f(u(x,t),v(x,t))$ and $g(u(x,t),v(x,t))$ are the reaction terms.

This paper is organized as follows. In Section 2, we derive an algorithm for calculating the normal form of Hopf bifurcation for the system (3). In Section 3, we obtain the normal form of Hopf bifurcation truncated to the third-order term by using our newly developed algorithm developed in Section 2, and the mathematical expressions of its corresponding coefficients are given. In Section 4, we consider the diffusive Holling-Tanner model with spatial memory and predator-taxis. The stability and Hopf bifurcation analysis of this model are studied, and some numerical simulations are also carried out. In Section 5, we give a brief conclusion and discussion.

2. Algorithm for calculating the normal form of Hopf bifurcation for the system (3)

2.1. Characteristic equation at the positive constant steady state

Define the real-valued Sobolev space

$$X:=\left\{(u,v)^T\in\left(W^{2,2}(0,\ell\pi)\right)^2:\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}=0 \text{ at } x=0,\ell\pi\right\}$$

with the inner product defined by

$$[U_1,U_2]=\int_0^{\ell\pi}U_1^TU_2\mathrm{d}x \text{ for } U_1=(u_1,v_1)^T\in X \text{ and } U_2=(u_2,v_2)^T\in X,$$

where the symbol T represents the transpose of vector, and let $\mathcal{C}:=C([-1,0],X)$ be the Banach space of continuous mappings from $[-1,0]$ to X with the sup norm. It is well known that the eigenvalue problem

$$\begin{cases}\tilde{\varphi}''(x)=\tilde{\lambda}\tilde{\varphi}(x),x\in(0,\ell\pi),\\ \tilde{\varphi}'(0)=\tilde{\varphi}'(\ell\pi)=0\end{cases}$$

has eigenvalues $\tilde{\lambda}_n=-n^2/\ell^2$ with corresponding normalized eigenfunctions

$$\beta_n^{(j)}=\gamma_n(x)e_j,\gamma_n(x)=\frac{\cos(nx/\ell)}{\|\cos(nx/\ell)\|_{L^2}}=\begin{cases}\dfrac{1}{\sqrt{\ell\pi}}, & n=0,\\ \dfrac{\sqrt{2}}{\sqrt{\ell\pi}}\cos\left(\dfrac{nx}{\ell}\right), & n\ge 1,\end{cases} \tag{4}$$

where $e_j,j=1,2$ is the unit coordinate vector of $\mathbb{R}^2$, and $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ is often called wave number, $\mathbb{N}_0$ is the set of all non-negative integers, $\mathbb{N}=\{1,2,\dots\}$ represents the set of all positive integers. Without loss of generality, we assume that $E_*(u_*,v_*)$ is the positive constant steady state of system (3). The linearized equation of (3) at $E_*(u_*,v_*)$ is

$$\begin{pmatrix}\dfrac{\partial u(x,t)}{\partial t}\\ \dfrac{\partial v(x,t)}{\partial t}\end{pmatrix}=D_1\begin{pmatrix}u_{xx}(x,t)\\ v_{xx}(x,t)\end{pmatrix}+D_2\begin{pmatrix}u_{xx}(x,t-\tau)\\ v_{xx}(x,t-\tau)\end{pmatrix}+A\begin{pmatrix}u(x,t)\\ v(x,t)\end{pmatrix} \tag{5}$$

where

$$D_1=\begin{pmatrix} d_{11} & \xi u_* \\ 0 & d_{22}\end{pmatrix}, D_2=\begin{pmatrix} 0 & 0 \\ -d_{21}v_* & 0\end{pmatrix}, A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} \tag{6}$$

and

$$a_{11}=\frac{\partial f(u_*,v_*)}{\partial u}, a_{12}=\frac{\partial f(u_*,v_*)}{\partial v}, a_{21}=\frac{\partial g(u_*,v_*)}{\partial u}, a_{22}=\frac{\partial g(u_*,v_*)}{\partial v}. \tag{7}$$

Therefore, the characteristic equation of (5) is

$$\prod_{n\in\mathbb{N}_0}\Gamma_n(\lambda)=0,$$

where $\Gamma_n(\lambda)=\det(\mathcal{M}_n(\lambda))$ with

$$\mathcal{M}_n(\lambda)=\lambda I_2+\frac{n^2}{\ell^2}D_1+\frac{n^2}{\ell^2}\mathrm{e}^{-\lambda\tau}D_2-A. \tag{8}$$

Here, det (.) represents the determinant of a matrix, I_2 is the identity matrix of 2×2, and D_1, D_2, A are defined by (6). Then we can obtain

$$\Gamma_n(\lambda)=\det\left(\mathcal{M}_n(\lambda)\right)=\lambda^2-T_n\lambda+\tilde{J}_n(\tau)=0, \tag{9}$$

where

$$\begin{aligned} T_n &= \mathrm{Tr}\,(A)-\mathrm{Tr}\,(D_1)\frac{n^2}{\ell^2}, \\ \tilde{J}_n(\tau) &= \left(d_{11}d_{22}+d_{21}\xi u_* v_*\mathrm{e}^{-\lambda\tau}\right)\frac{n^4}{\ell^4}-\left(d_{11}a_{22}+d_{22}a_{11}-a_{21}\xi u_*+d_{21}v_*a_{12}\mathrm{e}^{-\lambda\tau}\right)\frac{n^2}{\ell^2}+\mathrm{Det}\,(A)\end{aligned} \tag{10}$$

with $\mathrm{Tr}\,(A)=a_{11}+a_{22}$, $\mathrm{Tr}\,(D_1)=d_{11}+d_{22}$ and $\mathrm{Det}\,(A)=a_{11}a_{22}-a_{12}a_{21}$.

2.2. Basic assumption and equation transformation

Assumption 1. *Assume that at $\tau=\tau_c$, the characteristic Equation (9) has a pair of purely imaginary roots $\pm i\omega_{n_c}$ with $\omega_{n_c}>0$ for $n=n_c\in\mathbb{N}$, and all other roots of the characteristic Equation (9) have negative real parts. Let $\lambda(\tau)=\alpha_1(\tau)\pm i\alpha_2(\tau)$ be a pair of roots of the characteristic Equation (9) near $\tau=\tau_c$ satisfying $\alpha_1(\tau_c)=0$ and $\alpha_2(\tau_c)=\omega_{n_c}$. Meanwhile, the corresponding transversality condition holds.*

Let $\tau=\tau_c+\mu$ such that $\mu=0$ corresponds to the Hopf bifurcation value for system (3). Moreover, we shift $E_*(u_*,v_*)$ to the origin by setting

$$U(x,t)=\left(U_1(x,t),U_2(x,t)\right)^T=(u(x,t),v(x,t))^T-(u_*,v_*)^T,$$

and normalize the delay by rescaling the time variable $t\to t/\tau$. Furthermore, we rewrite $U(t)$ for $U(x,t)$, and $U_t\in\mathcal{C}$ for $U_t(\theta)=U(x,t+\theta)$, $-1\le\theta\le 0$. Then the system (3) becomes the compact form

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t}=d(\mu)\Delta(U_t)+L(\mu)(U_t)+F(U_t,\mu), \tag{11}$$

where for $\varphi=\left(\varphi^{(1)},\varphi^{(2)}\right)^T\in\mathcal{C}$, $d(\mu)\Delta$ is given by

$$d(\mu)\Delta(\varphi)=d_0\Delta(\varphi)+F^d(\varphi,\mu)$$

with

$$\begin{aligned} d_0\Delta(\varphi) &= \tau_c D_1\varphi_{xx}(0)+\tau_c D_2\varphi_{xx}(-1) \\ F^d(\varphi,\mu) &= \begin{pmatrix} \xi(\tau_c+\mu)\left(\varphi_x^{(1)}(0)\varphi_x^{(2)}(0)+\varphi^{(1)}(0)\varphi_{xx}^{(2)}(0)\right) \\ -d_{21}(\tau_c+\mu)\left(\varphi_x^{(1)}(-1)\varphi_x^{(2)}(0)+\varphi_{xx}^{(1)}(-1)\varphi^{(2)}(0)\right)\end{pmatrix}. \\ &\quad +\mu\begin{pmatrix} d_{11}\varphi_{xx}^{(1)}(0)+\xi u_*\varphi_{xx}^{(2)}(0) \\ -d_{21}v_*\varphi_{xx}^{(1)}(-1)+d_{22}\varphi_{xx}^{(2)}(0)\end{pmatrix}.\end{aligned} \tag{12}$$

Furthermore, $L(\mu):\mathcal{C}\to X$ is given by

$$L(\mu)(\varphi)=(\tau_c+\mu)A\varphi(0), \tag{13}$$

and $F:\mathcal{C}\times\mathbb{R}\to X$ is given by

$$F(\varphi,\mu)=(\tau_c+\mu)\begin{pmatrix} f\left(\varphi^{(1)}(0)+u_*,\varphi^{(2)}(0)+v_*\right) \\ g\left(\varphi^{(1)}(0)+u_*,\varphi^{(2)}(0)+v_*\right)\end{pmatrix}-L(\mu)(\varphi). \tag{14}$$

In what follows, we assume that $F(\varphi,\mu)$ is $C^k(k\geq 3)$ function, which is smooth with respect to φ and μ. Notice that μ is the perturbation parameter and is treated as a variable in the calculation of normal form. Moreover, by Equation (13), if we denote $L_0(\varphi)=\tau_c A\varphi(0)$, then Equation (11) can be rewritten as

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t}=d_0\Delta(U_t)+L_0(U_t)+\tilde{F}(U_t,\mu), \tag{15}$$

where the linear and nonlinear terms are separated, and

$$\tilde{F}(\varphi,\mu)=\mu A\varphi(0)+F(\varphi,\mu)+F^d(\varphi,\mu). \tag{16}$$

Thus, the linearized equation of (15) can be written as

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t}=d_0\Delta(U_t)+L_0(U_t). \tag{17}$$

Moreover, the characteristic equation for the linearized Equation (17) is

$$\prod_{n\in\mathbb{N}_0}\tilde{\Gamma}_n(\lambda)=0, \tag{18}$$

where $\widetilde{\Gamma}_n(\lambda)=\det\left(\widetilde{\mathcal{M}}_n(\lambda)\right)$ with

$$\widetilde{\mathcal{M}}_n(\lambda)=\lambda I_2+\tau_c\frac{n^2}{\ell^2}D_1+\tau_c\frac{n^2}{\ell^2}\mathrm{e}^{-\lambda}D_2-\tau_c A. \tag{19}$$

By comparing Equation (19) with Equation (8), we know that Equation (18) has a pair of purely imaginary roots $\pm\mathrm{i}\omega_c$ for $n=n_c\in\mathbb{N}$, and all other eigenvalues have negative real parts, where $\omega_c=\tau_c\omega_{n_c}$. In order to write Equation (15) as an abstract ordinary differential equation in a Banach space, follows by Faria[29], we can take the enlarged space

$$\mathcal{BC}:=\left\{\tilde{\psi}(\theta):[-1,0]\to X:\tilde{\psi}(\theta)\ \text{is continuous on}\ [-1,0),\exists\lim_{\theta\to 0^-}\tilde{\psi}(\theta)\in X\right\},$$

then Equation (15) is equivalent to an abstract ordinary differential equation on $\mathcal{BC}$

$$\frac{\mathrm{d}U_t}{\mathrm{d}t}=\tilde{A}U_t+X_0(\theta)\tilde{F}(U_t,\mu).$$

Here, $\tilde{A}$ is an operator from $\mathcal{C}_0^1=\{\varphi\in\mathcal{C}:\dot{\varphi}\in\mathcal{C},\varphi(0)\in\mathrm{dom}\,(\Delta)\}$ to $\mathcal{BC}$, which is defined by

$$\tilde{A}\varphi=\dot{\varphi}+X_0(\tau_c D_1\varphi_{xx}(0)+\tau_c D_2\varphi_{xx}(-1)+L_0(\varphi)-\dot{\varphi}(0)),$$

and $X_0(\theta)$ is given by

$$X_0(\theta)=\begin{cases}0, & -1\leq\theta<0,\\ I_2, & \theta=0.\end{cases}$$

In the following, the method given by Faria[29] is used to complete the decomposition of $\mathcal{BC}$. Let $C:=C\left([-1,0],\mathbb{R}^2\right),C^*:=C\left([0,1],\mathbb{R}^{2*}\right)$, where $\mathbb{R}^{2*}$ is the two-dimensional space of row vectors, and define the adjoint bilinear form on $C^*\times C$ as follows

$$\langle\hat{\psi}(s),\hat{\varphi}(\theta)\rangle_n=\hat{\psi}(0)\hat{\varphi}(0)-\int_{-1}^{0}\int_{0}^{\theta}\hat{\psi}(\xi-\theta)\mathrm{d}M_n(\theta)\hat{\varphi}(\xi)\mathrm{d}\xi$$

for $\hat{\psi}\in C^*,\hat{\varphi}\in C$ and $\xi\in[-1,0]$, where $M_n(\theta)$ is a bounded variation function from $[-1,0]$ to $\mathbb{R}^2\times\mathbb{R}^2$, i.e., $M_n(\theta)\in BV([-1,0],\mathbb{R}^2\times\mathbb{R}^2)$, such that for $\hat{\varphi}(\theta)\in C$, one has

$$-\tau_c\frac{n^2}{\ell^2}D_1\hat{\varphi}(0)-\tau_c\frac{n^2}{\ell^2}D_2\hat{\varphi}(-1)+L_0\left(\hat{\varphi}(\theta)\right)=\int_{-1}^{0}\mathrm{d}M_n(\theta)\hat{\varphi}(\theta).$$

By choosing

$$\Phi(\theta)=(\phi(\theta),\bar{\phi}(\theta)),\Psi(s)=\mathrm{col}\,(\psi^T(s),\bar{\psi}^T(s)),$$

where col(.) represents column vector, $\phi(\theta)=\mathrm{col}\,(\phi_1(\theta),\phi_2(\theta))=\phi\mathrm{e}^{\mathrm{i}\omega_c\theta}\in\mathbb{C}^2$ with $\phi=\mathrm{col}\,(\phi_1,\phi_2)$ is the eigenvector of Equation (17) associated with the eigenvalue $\mathrm{i}\omega_c$, and $\psi(s)=\mathrm{col}\,(\psi_1(s),\psi_2(s))=\psi\mathrm{e}^{-\mathrm{i}\omega_c s}\in\mathbb{C}^2$ with $\psi=\mathrm{col}\,(\psi_1,\psi_2)$ is the corresponding adjoint eigenvector such that

$$\langle \Psi(s), \Phi(\theta)\rangle_{n_c} = I_2,$$

where

$$\phi = \begin{pmatrix} 1 \\ \dfrac{a_{11} - \mathrm{i}\omega_{n_c} - d_{11}(n_c^2/\ell^2)}{\xi u_*(n_c^2/\ell^2) - a_{12}} \end{pmatrix}, \psi = \eta \begin{pmatrix} 1 \\ \dfrac{a_{12} - \xi u_*(n_c^2/\ell^2)}{\mathrm{i}\omega_{n_c} + d_{22}(n_c^2/\ell^2) - a_{22}} \end{pmatrix}$$

and

$$\eta = \frac{\mathrm{i}\omega_{n_c} + (n_c/\ell)^2 d_{22} - a_{22}}{2\mathrm{i}\omega_{n_c} + (n_c/\ell)^2 d_{11} - a_{11} + (n_c/\ell)^2 d_{22} - a_{22} + \tau_c a_{12} d_{21} v_* (n_c/\ell)^2 \mathrm{e}^{-\mathrm{i}\omega_c}}.$$

According to the method given by Faria[29], the phase space $\mathcal{C}$ can be decomposed as

$$\mathcal{C} = \mathcal{P} \oplus \mathcal{Q}, \mathcal{P} = \operatorname{Im}\pi, \mathcal{Q} = \operatorname{Ker}\pi,$$

where for $\tilde{\phi}(\theta) \in \mathcal{C}$, the projection $\pi: \mathcal{C} \to \mathcal{P}$ is defined by

$$\pi(\tilde{\phi}(\theta)) = \Phi(\theta) \left\langle \Psi(\theta), \begin{pmatrix} \left[\tilde{\phi}(\theta), \beta_{n_c}^{(1)}\right] \\ \left[\tilde{\phi}(\theta), \beta_{n_c}^{(2)}\right] \end{pmatrix} \right\rangle_{n_c} \gamma_{n_c}(x). \tag{20}$$

Therefore, by following the method given by Faria[29], $\mathcal{BC}$ can be divided into a direct sum of center subspace and its complementary space, that is

$$\mathcal{BC} = \mathcal{P} \oplus \operatorname{Ker} \pi, \tag{21}$$

where $\dim \mathcal{P} = 2$. It is easy to see that the projection π which is defined by (20), is extended to a continuous projection (which is still denoted by π), that is, $\pi: \mathcal{BC} \to \mathcal{P}$. In particular, for $\tilde{\alpha} \in X$, we have

$$\pi(X_0(\theta)\tilde{\alpha}) = \Phi(\theta)\Psi(0) \begin{pmatrix} \left[\tilde{\alpha}, \beta_{n_c}^{(1)}\right] \\ \left[\tilde{\alpha}, \beta_{n_c}^{(2)}\right] \end{pmatrix} \gamma_{n_c}(x). \tag{22}$$

By combining with Equations (20)–(22), $U_t(\theta)$ can be decomposed as

$$\begin{aligned} U_t(\theta) &= \Phi(\theta) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \gamma_{n_c}(x) + w = \left(z_1 \phi \mathrm{e}^{\mathrm{i}\omega_c \theta} + z_2 \bar{\phi} \mathrm{e}^{-\mathrm{i}\omega_c \theta}\right) \gamma_{n_c}(x) + w \\ &= \left(\phi(\theta) \ \ \bar{\phi}(\theta)\right) \begin{pmatrix} z_1 \gamma_{n_c}(x) \\ z_2 \gamma_{n_c}(x) \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \end{aligned} \tag{23}$$

where $w = \operatorname{col}(w_1, w_2)$ and

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left\langle \Psi(\theta), \begin{pmatrix} \left[U_t(\theta), \beta_{n_c}^{(1)}\right] \\ \left[U_t(\theta), \beta_{n_c}^{(2)}\right] \end{pmatrix} \right\rangle_{n_c} + \Psi(0) \begin{pmatrix} \left[U_t(\theta), \beta_{n_c}^{(1)}\right] \\ \left[U_t(\theta), \beta_{n_c}^{(2)}\right] \end{pmatrix}.$$

If we assume that

$$\Phi(\theta) = \left(\phi(\theta), \bar{\phi}(\theta)\right), \ z_x = \left(z_1 \gamma_{n_c}(x), z_2 \gamma_{n_c}(x)\right)^T,$$

then (23) can be rewritten as

$$U_t(\theta) = \Phi(\theta) z_x + w \ \text{ with } \ w \in \mathcal{C}_0^1 \cap \operatorname{Ker}\pi := \mathcal{Q}^1. \tag{24}$$

Then by combining with Equation (24), the system (15) is decomposed as a system of abstract ordinary differential equations (ODEs) on $\mathbb{R}^2 \times \operatorname{Ker} \pi$, with finite and infinite dimensional variables are separated in the linear term. That is

$$\begin{cases} \dot{z} = Bz + \Psi(0) \begin{pmatrix} \left[\tilde{F}(\Phi(\theta) z_x + w, \mu), \beta_{n_c}^{(1)}\right] \\ \left[\tilde{F}(\Phi(\theta) z_x + w, \mu), \beta_{n_c}^{(2)}\right] \end{pmatrix}, \\ \dot{w} = A_{\mathcal{Q}^1} w + (I_2 - \pi) X_0(\theta) \tilde{F}(\Phi(\theta) z_x + w, \mu), \end{cases} \tag{25}$$

where $z = (z_1, z_2)^T, B = \operatorname{diag}\{\mathrm{i}\omega_c, -\mathrm{i}\omega_c\}$ is the diagonal matrix, and $A_{\mathcal{Q}^1}: \mathcal{Q}^1 \to \operatorname{Ker} \pi$ is defined by

$$A_{\mathcal{Q}^1} w = \dot{w} + X_0(\theta)\left(\tau_c D_1 w_{xx}(0) + \tau_c D_2 w_{xx}(-1) + L_0(w) - \dot{w}(0)\right).$$

Consider the formal Taylor expansions

$$\tilde{F}(\varphi,\mu)=\sum_{j\geq 2}\frac{1}{j!}\tilde{F}_j(\varphi,\mu), F(\varphi,\mu)=\sum_{j\geq 2}\frac{1}{j!}F_j(\varphi,\mu), F^d(\varphi,\mu)=\sum_{j\geq 2}\frac{1}{j!}F_j^d(\varphi,\mu).$$

From Equation (16), we have

$$\tilde{F}_2(\varphi,\mu)=2\mu A\varphi(0)+F_2(\varphi,\mu)+F_2^d(\varphi,\mu) \tag{26}$$

and

$$\tilde{F}_j(\varphi,\mu)=F_j(\varphi,\mu)+F_j^d(\varphi,\mu), j=3,4,\cdots. \tag{27}$$

By combining with Equation (22), the system (25) can be rewritten as

$$\begin{cases}\dot{z}=Bz+\sum_{j\geq 2}\frac{1}{j!}f_j^1(z,w,\mu),\\ \dot{w}=A_{Q^1}w+\sum_{j\geq 2}\frac{1}{j!}f_j^2(z,w,\mu),\end{cases}$$

where

$$\begin{aligned}f_j^1(z,w,\mu)&=\Psi(0)\begin{pmatrix}\left[\tilde{F}_j(\Phi(\theta)z_x+w,\mu),\beta_{n_c}^{(1)}\right]\\ \left[\tilde{F}_j(\Phi(\theta)z_x+w,\mu),\beta_{n_c}^{(2)}\right]\end{pmatrix},\\ f_j^2(z,w,\mu)&=(I_2-\pi)X_0(\theta)\tilde{F}_j(\Phi(\theta)z_x+w,\mu).\end{aligned} \tag{28}$$

In terms of the normal form theory of retarded functional differential equations with parameters[30], after a recursive transformation of variables of the form

$$(z,w)=(\tilde{z},\tilde{w})+\frac{1}{j!}\left(U_j^1(\tilde{z},\mu),U_j^2(\tilde{z},\mu)(\theta)\right), j\geq 2, \tag{29}$$

where $z,\tilde{z}\in\mathbb{R}^2, w,\tilde{w}\in Q^1$ and $U_j^1:\mathbb{R}^3\to\mathbb{R}^2, U_j^2:\mathbb{R}^3\to Q^1$ are homogeneous polynomials of degree j in $\tilde{z}$ and μ, a locally center manifold for Equation (15) satisfies $w=0$ and the flow on it is given by the two-dimensional ODEs

$$\dot{z}=Bz+\sum_{j\geq 2}\frac{1}{j!}g_j^1(z,0,\mu),$$

which is the normal form as in the usual sense for ODEs. By following study[29,30], we have

$$g_2^1(z,0,\mu)=\mathrm{Proj}_{\mathrm{Ker}\,(M_2^1)}\, f_2^1(z,0,\mu) \tag{30}$$

and

$$g_3^1(z,0,\mu)=\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\,\tilde{f}_3^1(z,0,\mu)=\mathrm{Proj}_S\,\tilde{f}_3^1(z,0,0)+O(\mu^2|z|), \tag{31}$$

where $\mathrm{Proj}_p\ (q)$ represents the projection of q on p, and $\tilde{f}_3^1(z,0,\mu)$ is vector and its element is the cubic polynomial of (z,μ) after the variable transformation of (29), and it is determined by (41),

$$\begin{aligned}\mathrm{Ker}\,(M_2^1)&=\mathrm{Span}\left\{\begin{pmatrix}\mu z_1\\0\end{pmatrix},\begin{pmatrix}0\\ \mu z_2\end{pmatrix}\right\},\\ \mathrm{Ker}(M_3^1)&=\mathrm{Span}\left\{\begin{pmatrix}z_1^2z_2\\0\end{pmatrix},\begin{pmatrix}\mu^2z_1\\0\end{pmatrix},\begin{pmatrix}0\\ z_1z_2^2\end{pmatrix},\begin{pmatrix}0\\ \mu^2z_2\end{pmatrix}\right\},\end{aligned} \tag{32}$$

and

$$S=\mathrm{Span}\left\{\begin{pmatrix}z_1^2z_2\\0\end{pmatrix},\begin{pmatrix}0\\ z_1z_2^2\end{pmatrix}\right\}. \tag{33}$$

In the following, for notational convenience, we let

$$\mathcal{H}\left(\alpha z_1^{q_1}z_2^{q_2}\mu\right)=\begin{pmatrix}\alpha z_1^{q_1}z_2^{q_2}\mu\\ \bar{\alpha}z_1^{q_2}z_2^{q_1}\mu\end{pmatrix}, \alpha\in\mathbb{C}.$$

We then calculate $g_j^1(z,0,\mu), j=2,3$ step by step.

2.3. Algorithm for calculating the normal form of Hopf bifurcation

2.3.1. Calculation of $g_2^1(z,0,\mu)$

From the second mathematical expression in (12), we have

$$F_2^d(\varphi,\mu) = F_{20}^d(\varphi) + \mu F_{21}^d(\varphi) \tag{34}$$

and

$$F_3^d(\varphi,\mu) = \mu F_{31}^d(\varphi), F_j^d(\varphi,\mu) = (0,0)^T, j = 4,5,\cdots, \tag{35}$$

where

$$\begin{cases} F_{20}^d(\varphi) = 2\begin{pmatrix} \xi\tau_c\left(\varphi_x^{(1)}(0)\varphi_x^{(2)}(0) + \varphi^{(1)}(0)\varphi_{xx}^{(2)}(0)\right) \\ -d_{21}\tau_c\left(\varphi_x^{(1)}(-1)\varphi_x^{(2)}(0) + \varphi_{xx}^{(1)}(-1)\varphi^{(2)}(0)\right)\end{pmatrix}, \\ F_{21}^d(\varphi) = 2D_1\varphi_{xx}(0) + 2D_2\varphi_{xx}(-1), \\ F_{31}^d(\varphi) = 6\begin{pmatrix} \xi\left(\varphi_x^{(1)}(0)\varphi_x^{(2)}(0) + \varphi^{(1)}(0)\varphi_{xx}^{(2)}(0)\right) \\ -d_{21}\left(\varphi_x^{(1)}(-1)\varphi_x^{(2)}(0) + \varphi_{xx}^{(1)}(-1)\varphi^{(2)}(0)\right)\end{pmatrix}. \end{cases} \tag{36}$$

Furthermore, it is easy to verify that

$$\begin{aligned} &\begin{pmatrix} \left[2\mu A(\Phi(0)z_x), \beta_{n_c}^{(1)}\right] \\ \left[2\mu A(\Phi(0)z_x), \beta_{n_c}^{(2)}\right]\end{pmatrix} = 2\mu A\left(\Phi(0)\begin{pmatrix} z_1 \\ z_2\end{pmatrix}\right), \\ &\begin{pmatrix} \left[\mu F_{21}^d(\Phi(\theta)z_x), \beta_{n_c}^{(1)}\right] \\ \left[\mu F_{21}^d(\Phi(\theta)z_x), \beta_{n_c}^{(2)}\right]\end{pmatrix} = -\frac{2n_c^2}{\ell^2}\mu\left(D_1\left(\Phi(0)\begin{pmatrix} z_1 \\ z_2\end{pmatrix}\right) + D_2\left(\Phi(-1)\begin{pmatrix} z_1 \\ z_2\end{pmatrix}\right)\right). \end{aligned} \tag{37}$$

From Equation (14), we have $F_2(\Phi(\theta)z_x,\mu) = F_2(\Phi(\theta)z_x,0)$ for all $\mu \in \mathbb{R}$. It follows from the first mathematical expression in (28) that

$$f_2^1(z,0,\mu) = \Psi(0)\begin{pmatrix} \left[\tilde{F}_2(\Phi(\theta)z_x,\mu), \beta_{n_c}^{(1)}\right] \\ \left[\tilde{F}_2(\Phi(\theta)z_x,\mu), \beta_{n_c}^{(2)}\right]\end{pmatrix}.$$

This, together with Equation (26), (30), (32) and (34)–(37), yields to

$$g_2^1(z,0,\mu) = \mathrm{Proj}_{\mathrm{Ker}(M_2^1)} f_2^1(z,0,\mu) = \mathcal{H}(B_1\mu z_1), \tag{38}$$

where

$$B_1 = 2\psi^T(0)\left(A\phi(0) - \frac{n_c^2}{\ell^2}\left(D_1\phi(0) + D_2\phi(-1)\right)\right). \tag{39}$$

2.3.2. Calculation of $g_3^1(z,0,\mu)$

In this subsection, we calculate the third term $g_3^1(z,0,\mu)$ in terms of Equation (31). Denote

$$\begin{aligned} f_2^{(1,1)}(z,w,0) &= \Psi(0)\begin{pmatrix} \left[F_2(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(1)}\right] \\ \left[F_2(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(2)}\right]\end{pmatrix}, \\ f_2^{(1,2)}(z,w,0) &= \Psi(0)\begin{pmatrix} \left[F_2^d(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(1)}\right] \\ \left[F_2^d(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(2)}\right]\end{pmatrix}. \end{aligned} \tag{40}$$

It follows from Equation (38) that $g_2^1(z,0,0) = (0,0)^T$. Then $\tilde{f}_3^1(z,0,0)$ is determined by

$$\tilde{f}_3^1(z,0,0) = f_3^1(z,0,0) + \frac{3}{2}\Big((D_z f_2^1(z,0,0))U_2^1(z,0) + \left(D_w f_2^{(1,1)}(z,0,0)\right) U_2^2(z,0)(\theta) + \left(D_{w,w_x,w_{xx}} f_2^{(1,2)}(z,0,0)\right) U_2^{(2,d)}(z,0)(\theta)\Big), \tag{41}$$

where $f_2^1(z,0,0) = f_2^{(1,1)}(z,0,0) + f_2^{(1,2)}(z,0,0)$,

$$\begin{aligned} D_{w,w_x,w_{xx}} f_2^{(1,2)}(z,0,0) &= \left(D_w f_2^{(1,2)}(z,0,0), D_{w_x} f_2^{(1,2)}(z,0,0), D_{w_{xx}} f_2^{(1,2)}(z,0,0)\right), \\ U_2^1(z,0) &= (M_2^1)^{-1}\mathrm{Proj}_{\mathrm{Im}\,(M_2^1)}\ f_2^1(z,0,0), U_2^2(z,0)(\theta) = (M_2^2)^{-1} f_2^2(z,0,0), \end{aligned} \tag{42}$$

and

$$U_2^{(2,d)}(z,0)(\theta) = \mathrm{col}\left(U_2^2(z,0)(\theta), U_{2,x}^2(z,0)(\theta), U_{2,xx}^2(z,0)(\theta)\right). \tag{43}$$

According to Equation (41), we calculate $\mathrm{Proj}_S\ \tilde{f}_3^1(z,0,0)$ by the following four steps.

Step 1 Calculation of $\mathrm{Proj}_S\ f_3^1(z,0,0)$:

Writing $F_3(\Phi(\theta)z_x,0)$ as follows

$$F_3(\Phi(\theta)z_x,0) = \sum_{q_1+q_2=3} A_{q_1q_2} z_1^{q_1} z_2^{q_2} \gamma_{n_c}^3(x), \tag{44}$$

where $A_{q_1q_2} = \bar{A}_{q_2q_1}$ with $q_1, q_2 \in \mathbb{N}_0$. From Equations (27) and (35), we have $\tilde{F}_3(\Phi(\theta)z_x,0) = F_3(\Phi(\theta)z_x,0)$, and thus

$$\mathrm{Proj}_S\, f_3^1(z,0,0) = \mathcal{H}(B_{21} z_1^2 z_2),$$

where

$$B_{21} = \frac{3}{2\ell\pi}\psi^T A_{21}. \tag{45}$$

Step 2 Calculation of $\mathrm{Proj}_S\ \left(\left(D_z f_2^1(z,0,0)\right)U_2^1(z,0)\right)$:

Form Equations (26) and (34), we have

$$\tilde{F}_2(\Phi(\theta)z_x,0) = F_2(\Phi(\theta)z_x,0) + F_{20}^d(\Phi(\theta)z_x). \tag{46}$$

From Equation (14), we write

$$\begin{aligned} F_2(\Phi(\theta)z_x + w,\mu) &= F_2(\Phi(\theta)z_x + w,0) \\ &= \sum_{q_1+q_2=2} A_{q_1q_2} z_1^{q_1} z_2^{q_2} \gamma_{n_c}^2(x) + S_2(\Phi(\theta)z_x,w) + O(|w|^2), \end{aligned} \tag{47}$$

where $S_2(\Phi(\theta)z_x,w)$ is the product term of $\Phi(\theta)z_x$ and w. By combining with Equations (34) and (36), we write

$$F_2^d(\Phi(\theta)z_x,0) = F_{20}^d(\Phi(\theta)z_x) = \sum_{q_1+q_2=2} A_{q_1q_2}^d z_1^{q_1} z_2^{q_2}\left(\xi_{n_c}^2(x) - \gamma_{n_c}^2(x)\right)\frac{n_c^2}{\ell^2}, \tag{48}$$

where $\xi_{n_c}(x) = (\sqrt{2}/\sqrt{\ell\pi})\sin\left((n_c/\ell)x\right)$, and

$$\begin{cases} A_{20}^d = \begin{pmatrix} 2\xi\tau_c\phi_1(0)\phi_2(0) \\ -2d_{21}\tau_c\phi_1(-1)\phi_2(0) \end{pmatrix} = \bar{A}_{02}^d, \\ A_{11}^d = \begin{pmatrix} 4\xi\tau_c \mathrm{Re}\,\{\phi_1(0)\bar{\phi}_2(0)\} \\ -4d_{21}\tau_c\mathrm{Re}\,\{\phi_1(-1)\bar{\phi}_2(0)\} \end{pmatrix}. \end{cases} \tag{49}$$

From $\xi_{n_c}(x) = (\sqrt{2}/\sqrt{\ell\pi})\sin\left((n_c/\ell)x\right)$ and Equation (4), it is easy to verify that

$$\int_0^{\ell\pi} \gamma_{n_c}^3(x)\mathrm{d}x = \int_0^{\ell\pi} \xi_{n_2}^2(x)\gamma_{n_c}(x)\mathrm{d}x = 0.$$

Then by combining with Equations (46)–(48), we have

$$f_2^1(z,0,0)=\Psi(0)\begin{pmatrix}\left[\tilde{F}_2(\Phi(\theta)z_x,0),\beta_{n_c}^{(1)}\right]\\ \left[\tilde{F}_2(\Phi(\theta)z_x,0),\beta_{n_c}^{(2)}\right]\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \tag{50}$$

Thus, by combining with Equations (33) and (50), we have

$$\mathrm{Proj}_S\left(\left(D_z f_2^1(z,0,0)\right)U_2^1(z,0)\right)=\begin{pmatrix}0\\0\end{pmatrix}.$$

Step 3 Calculation of Proj$_S$ $\left(\left(D_w f_2^{(1,1)}(z,0,0)\right)U_2^2(z,0)(\theta)\right)$:

Let

$$U_2^2(z,0)(\theta)\triangleq h(\theta,z)=\sum_{n\in\mathbb{N}_0}h_n(\theta,z)\gamma_n(x), \tag{51}$$

where

$$h_n(\theta,z)=\textstyle\sum_{q_1+q_2=2}h_{n,q_1q_2}(\theta)z_1^{q_1}z_2^{q_2}.$$

Then we have

$$\begin{pmatrix}\left[S_2\left(\Phi(\theta)z_x,\sum_{n\in\mathbb{N}_0}h_n(\theta,z)\gamma_n(x)\right),\beta_{n_c}^{(1)}\right]\\ \left[S_2\left(\Phi(\theta)z_x,\sum_{n\in\mathbb{N}_0}h_n(\theta,z)\gamma_n(x)\right),\beta_{n_c}^{(2)}\right]\end{pmatrix}$$
$$=\sum_{n\in\mathbb{N}_0}b_n\left(S_2(\phi(\theta)z_1,h_n(\theta,z))+S_2(\bar{\phi}(\theta)z_2,h_n(\theta,z))\right),$$

where

$$b_n=\int_0^{\ell\pi}\gamma_{n_c}^2(x)\gamma_n(x)\mathrm{d}x=\begin{cases}\dfrac{1}{\sqrt{\ell\pi}}, & n=0,\\ \dfrac{1}{\sqrt{2\ell\pi}}, & n=2n_c,\\ 0, & \text{otherwise.}\end{cases} \tag{52}$$

Hence, we have

$$\left(D_w f_2^{(1,1)}(z,0,0)\right)U_2^2(z,0)(\theta)$$
$$=\Psi(0)\left(\sum_{n=0,2n_c}b_n\left(S_2\big(\phi(\theta)z_1,h_n(\theta,z)\big)+S_2\left(\bar{\phi}(\theta)z_2,h_n(\theta,z)\right)\right)\right),$$

and

$$\mathrm{Proj}_S\left(\left(D_w f_2^{(1,1)}(z,0,0)\right)U_2^2(z,0)(\theta)\right)=\mathcal{H}(B_{22}z_1^2z_2),$$

where

$$\begin{aligned}B_{22}\ &=\frac{1}{\sqrt{\ell\pi}}\psi^T\left(S_2\big(\phi(\theta),h_{0,11}(\theta)\big)+S_2\big(\bar{\phi}(\theta),h_{0,20}(\theta)\big)\right)\\ &\quad+\frac{1}{\sqrt{2\ell\pi}}\psi^T\left(S_2\big(\phi(\theta),h_{2n_c,11}(\theta)\big)+S_2\big(\bar{\phi}(\theta),h_{2n_c,20}(\theta)\big)\right).\end{aligned} \tag{53}$$

Step 4 Calculation of Proj$_S$ $\left(\left(D_{w,w_x,w_{xx}} f_2^{(1,2)}(z,0,0)\right)U_2^{(2,d)}(z,0)(\theta)\right)$:

Denote $\varphi(\theta)=\left(\varphi^{(1)}(\theta),\varphi^{(2)}(\theta)\right)^T=\Phi(\theta)z_x$,

$$F_2^d(\varphi(\theta), w, w_x, w_{xx}) = F_2^d(\varphi(\theta) + w, 0) = F_{20}^d(\varphi(\theta) + w)$$
$$= 2\begin{pmatrix} \xi\tau_c\left(\left(\varphi_x^{(1)}(0) + (w_1)_x(0)\right)\left(\varphi_x^{(2)}(0) + (w_2)_x(0)\right) + \left(\varphi^{(1)}(0) + w_1(0)\right)\left(\varphi_{xx}^{(2)}(0) + (w_2)_{xx}(0)\right)\right) \\ -d_{21}\tau_c\begin{pmatrix}\left(\varphi_x^{(1)}(-1) + (w_1)_x(-1)\right)\left(\varphi_x^{(2)}(0) + (w_2)_x(0)\right) + \\ \left(\varphi_{xx}^{(1)}(-1) + (w_1)_{xx}(-1)\right)\left(\varphi^{(2)}(0) + w_2(0)\right)\end{pmatrix}\end{pmatrix}$$

and

$$\tilde{S}_2^{(d,1)}(\varphi(\theta), w) = 2\begin{pmatrix} \xi\tau_c\varphi_{xx}^{(2)}(0)w_1(0) \\ -d_{21}\tau_c\varphi_{xx}^{(1)}(-1)w_2(0)\end{pmatrix},$$
$$\tilde{S}_2^{(d,2)}(\varphi(\theta), w_x) = 2\begin{pmatrix} \xi\tau_c\left(\varphi_x^{(2)}(0)(w_1)_x(0) + \varphi_x^{(1)}(0)(w_2)_x(0)\right) \\ -d_{21}\tau_c\left(\varphi_x^{(2)}(0)(w_1)_x(-1) + \varphi_x^{(1)}(-1)(w_2)_x(0)\right)\end{pmatrix},$$
$$\tilde{S}_2^{(d,3)}(\varphi(\theta), w_{xx}) = 2\begin{pmatrix} \xi\tau_c\varphi^{(1)}(0)(w_2)_{xx}(0) \\ -d_{21}\tau_c\varphi^{(2)}(0)(w_1)_{xx}(-1)\end{pmatrix}.$$

By combining with Equations (4) and (51), we have

$$\begin{cases} U_{2,x}^2(z,0)(\theta) = h_x(\theta, z) = -\sum_{n\in\mathbb{N}_0} h_n(\theta, z)\xi_n(x)\frac{n}{\ell}, \\ U_{2,xx}^2(z,0)(\theta) = h_{xx}(\theta, z) = -\sum_{n\in\mathbb{N}_0} h_n(\theta, z)\gamma_n(x)\frac{n^2}{\ell^2}.\end{cases}$$

Then we have

$$\left(D_{w,w_x,w_{xx}}F_2^d(\varphi(\theta), w, w_x, w_{xx})\right)U_2^{(2,d)}(z,0)(\theta)$$
$$= \tilde{S}_2^{(d,1)}(\varphi(\theta), h(\theta, z)) + \tilde{S}_2^{(d,2)}(\varphi(\theta), h_x(\theta, z)) + \tilde{S}_2^{(d,3)}(\varphi(\theta), h_{xx}(\theta, z))$$

and

$$\begin{pmatrix}\left[\tilde{S}_2^{(d,1)}(\varphi(\theta), h(\theta, z)), \beta_{n_c}^{(1)}\right] \\ \left[\tilde{S}_2^{(d,1)}(\varphi(\theta), h(\theta, z)), \beta_{n_c}^{(2)}\right]\end{pmatrix}$$
$$= -(n_c/\ell)^2\sum_{n\in\mathbb{N}_0} b_n\left(S_2^{(d,1)}(\phi(\theta)z_1, h_n(\theta, z)) + S_2^{(d,1)}(\bar{\phi}(\theta)z_2, h_n(\theta, z))\right),$$
$$\begin{pmatrix}\left[\tilde{S}_2^{(d,2)}(\varphi(\theta), h_x(\theta, z)), \beta_{n_c}^{(1)}\right] \\ \left[\tilde{S}_2^{(d,2)}(\varphi(\theta), h_x(\theta, z)), \beta_{n_c}^{(2)}\right]\end{pmatrix}$$
$$= (n_c/\ell)\sum_{n\in\mathbb{N}_0} (n/\ell)c_n\left(S_2^{(d,2)}(\phi(\theta)z_1, h_n(\theta, z)) + S_2^{(d,2)}(\bar{\phi}(\theta)z_2, h_n(\theta, z))\right),$$
$$\begin{pmatrix}\left[\tilde{S}_2^{(d,3)}(\varphi(\theta), h_{xx}(\theta, z)), \beta_{n_c}^{(1)}\right] \\ \left[\tilde{S}_2^{(d,3)}(\varphi(\theta), h_{xx}(\theta, z)), \beta_{n_c}^{(2)}\right]\end{pmatrix}$$
$$= -\sum_{n\in\mathbb{N}_0} (n/\ell)^2 b_n\left(S_2^{(d,3)}(\phi(\theta)z_1, h_n(\theta, z)) + S_2^{(d,3)}(\bar{\phi}(\theta)z_2, h_n(\theta, z))\right),$$

where b_n is defined by Equation (52) and

$$c_n = \int_0^{\ell\pi} \xi_{n_c}(x)\xi_n(x)\gamma_{n_c}(x)\mathrm{d}x = \begin{cases}\frac{1}{\sqrt{2\ell\pi}}, & n = 2n_c, \\ 0, & \text{otherwise},\end{cases}$$

and for

$$\phi(\theta) = \left(\phi_1(\theta), \phi_2(\theta)\right)^T, y(\theta) = \left(y_1(\theta), y_2(\theta)\right)^T \in C([-1,0], \mathbb{R}^2),$$

we have

$$\begin{cases} S_2^{(d,1)}\big(\phi(\theta), y(\theta)\big) = 2\begin{pmatrix} \xi\tau_c\phi_2(0)y_1(0) \\ -d_{21}\tau_c\phi_1(-1)y_2(0) \end{pmatrix}, \\ S_2^{(d,2)}\big(\phi(\theta), y(\theta)\big) = 2\begin{pmatrix} \xi\tau_c\big(\phi_2(0)y_1(0) + \phi_1(0)y_2(0)\big) \\ -d_{21}\tau_c\big(\phi_2(0)y_1(-1) + \phi_1(-1)y_2(0)\big) \end{pmatrix}, \\ S_2^{(d,3)}\big(\phi(\theta), y(\theta)\big) = 2\begin{pmatrix} \xi\tau_c\phi_1(0)y_2(0) \\ -d_{21}\tau_c\phi_2(0)y_1(-1) \end{pmatrix}. \end{cases}$$

Furthermore, by combining with Equations (40), (42) and (43), we have

$$\begin{aligned} &\left(D_{w,w_x,w_{xx}} f_2^{(1,2)}(z,0,0)\right) U_2^{(2,d)}(z,0)(\theta) \\ &= \Psi(0)\begin{pmatrix} \left[D_{w,w_x,w_{xx}} F_2^d(\varphi(\theta), w, w_x, w_{xx}) U_2^{(2,d)}(z,0)(\theta), \beta_{n_c}^{(1)}\right] \\ \left[D_{w,w_x,w_{xx}} F_2^d(\varphi(\theta), w, w_x, w_{xx}) U_2^{(2,d)}(z,0)(\theta), \beta_{n_c}^{(2)}\right] \end{pmatrix}, \end{aligned}$$

and then we obtain

$$\mathrm{Proj}_S\left(\left(D_{w,w_x,w_{xx}} f_2^{(1,2)}(z,0,0)\right) U_2^{(2,d)}(z,0)(\theta)\right) = \mathcal{H}(B_{23} z_1^2 z_2),$$

where

$$\begin{aligned} B_{23} = \; & -\frac{1}{\sqrt{\ell\pi}}(n_c/\ell)^2\psi^T\left(S_2^{(d,1)}\big(\phi(\theta), h_{0,11}(\theta)\big) + S_2^{(d,1)}\big(\bar{\phi}(\theta), h_{0,20}(\theta)\big)\right) \\ & + \frac{1}{\sqrt{2\ell\pi}}\psi^T \sum_{j=1,2,3} b_{2n_c}^{(j)}\left(S_2^{(d,j)}\big(\phi(\theta), h_{2n_c,11}(\theta)\big) + S_2^{(d,j)}\big(\bar{\phi}(\theta), h_{2n_c,20}(\theta)\big)\right) \end{aligned} \tag{54}$$

with

$$b_{2n_c}^{(1)} = -\frac{n_c^2}{\ell^2}, b_{2n_c}^{(2)} = \frac{2n_c^2}{\ell^2}, b_{2n_c}^{(3)} = -\frac{(2n_c)^2}{\ell^2}.$$

3. Normal form of Hopf bifurcation for the system (3)

By using the algorithm developed in Section 2, we can obtain the normal form of Hopf bifurcation for the system (3) truncated to the third-order term

$$\dot{z} = Bz + \frac{1}{2}\begin{pmatrix} B_1 z_1 \mu \\ \bar{B}_1 z_2 \mu \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} B_2 z_1^2 z_2 \\ \bar{B}_2 z_1 z_2^2 \end{pmatrix} + O(|z|\mu^2 + |z|^4), \tag{55}$$

where

$$\begin{aligned} B_1 &= 2\psi^T(0)\left(A\phi(0) - \frac{n_c^2}{\ell^2}\big(D_1\phi(0) + D_2\phi(-1)\big)\right), \\ B_2 &= B_{21} + \frac{3}{2}(B_{22} + B_{23}). \end{aligned} \tag{56}$$

Here, B_1 is determined by Equation (39), B_{21}, B_{22} and B_{23} are determined by Equations (45), (53), (54), respectively, and they can be calculated by using the MATLAB software. The normal form Equation (55) can be written in real coordinates through the change of variables $z_1 = v_1 - \mathrm{i}v_2, z_2 = v_1 + \mathrm{i}v_2$, and then changing to polar coordinates by $v_1 = \rho\cos\Theta, v_2 = \rho\sin\Theta$, where Θ is the azimuthal angle. Therefore, by the above transformations and removing the azimuthal term Θ, Equation (55) can be rewritten as

$$\dot{\rho} = K_1\mu\rho + K_2\rho^3 + O(\mu^2\rho + |(\rho,\mu)|^4),$$

where

$$K_1 = \frac{1}{2}\mathrm{Re}(B_1), K_2 = \frac{1}{3!}\mathrm{Re}(B_2).$$

According to the study[31], the sign of K_1K_2 determines the direction of the Hopf bifurcation, and the sign of K_2 determines the stability of the Hopf bifurcation periodic solution. More precisely, we have the following results:

(i) when $K_1K_2 < 0$, the Hopf bifurcation is supercritical, and the Hopf bifurcation periodic solution is stable for $K_2 < 0$ and unstable for $K_2 > 0$;

(ii) when $K_1K_2 > 0$, the Hopf bifurcation is subcritical, and the Hopf bifurcation periodic solution is stable for $K_2 < 0$ and unstable for $K_2 > 0$.

By combining with (53), (54) and the second mathematical expression in Equation (56), it is obvious that in order to obtain the value of K_2, we still need to calculate $h_{0,20}(\theta), h_{0,11}(\theta), h_{2n_c,20}(\theta), h_{2n_c,11}(\theta)$ and $A_{q_1q_2}$.

3.1. Calculations of $h_{0,20}(\theta), h_{0,11}(\theta), h_{2n_c,20}(\theta)$ and $h_{2n_c,11}(\theta)$

From the study[29], we have

$$M_2^2(h_n(\theta,z)\gamma_n(x)) = D_z(h_n(\theta,z)\gamma_n(x))Bz - A_{Q^1}\big(h_n(\theta,z)\gamma_n(x)\big),$$

which leads to

$$\begin{aligned}&\begin{pmatrix}\left[M_2^2(h_n(\theta,z)\gamma_n(x)),\beta_n^{(1)}\right]\\ \left[M_2^2(h_n(\theta,z)\gamma_n(x)),\beta_n^{(2)}\right]\end{pmatrix}\\ &= 2\mathrm{i}\omega_c\big(h_{n,20}(\theta)z_1^2 - h_{n,02}(\theta)z_2^2\big) - \Big(\dot h_n(\theta,z) + X_0(\theta)\Big(\mathcal{L}_0\big(h_n(\theta,z)\big) - \dot h_n(0,z)\Big)\Big),\end{aligned} \tag{57}$$

where

$$\mathcal{L}_0(h_n(\theta,z)) = -\tau_c(n/\ell)^2(D_1h_n(0,z) + D_2h_n(-1,z)) + \tau_cAh_n(0,z).$$

By Equation (22) and the second mathematical expression in (28), we have

$$\begin{aligned}f_2^2(z,0,0) &= X_0(\theta)\tilde F_2(\Phi(\theta)z_x,0) - \pi\Big(X_0(\theta)\tilde F_2(\Phi(\theta)z_x,0)\Big)\\ &= X_0(\theta)\tilde F_2(\Phi(\theta)z_x,0) - \Phi(\theta)\Psi(0)\begin{pmatrix}\left[\tilde F_2(\Phi(\theta)z_x,0),\beta_{n_c}^{(1)}\right]\\ \left[\tilde F_2(\Phi(\theta)z_x,0),\beta_{n_c}^{(2)}\right]\end{pmatrix}\gamma_{n_c}(x).\end{aligned} \tag{58}$$

Furthermore, by Equations (46)–(48), we have

$$\begin{pmatrix}\left[f_2^2(z,0,0),\beta_n^{(1)}\right]\\ \left[f_2^2(z,0,0),\beta_n^{(2)}\right]\end{pmatrix} = \begin{cases}\dfrac{1}{\sqrt{\ell\pi}}X_0(\theta)(A_{20}z_1^2 + A_{02}z_2^2 + A_{11}z_1z_2), & n = 0,\\ \dfrac{1}{\sqrt{2\ell\pi}}X_0(\theta)\big(\tilde A_{20}z_1^2 + \tilde A_{02}z_2^2 + \tilde A_{11}z_1z_2\big), & n = 2n_c,\end{cases} \tag{59}$$

where $\tilde A_{j_1j_2}$ is defined as follows

$$\begin{cases}\tilde A_{j_1j_2} = A_{j_1j_2} - 2(n_c/\ell)^2A_{j_1j_2}^d,\\ j_1,j_2 = 0,1,2, j_1 + j_2 = 2,\end{cases} \tag{60}$$

where $A_{j_1j_2}^d$ is determined by Equation (49), and $A_{j_1j_2}$ will be calculated in the following section. Therefore, by Equations (42), (57), (58), (59), and by matching the coefficients of z_1^2 and z_1z_2, we have

$$n = 0,\begin{cases}z_1^2:\begin{cases}\dot h_{0,20}(\theta) - 2\mathrm{i}\omega_ch_{0,20}(\theta) = (0,0)^T,\\ \dot h_{0,20}(0) - L_0\Big(h_{0,20}(\theta)\Big) = \dfrac{1}{\sqrt{\ell\pi}}A_{20},\end{cases}\\ z_1z_2:\begin{cases}\dot h_{0,11}(\theta) = (0,0)^T,\\ \dot h_{0,11}(0) - L_0\big(h_{0,11}(\theta)\big) = \dfrac{1}{\sqrt{\ell\pi}}A_{11}\end{cases}\end{cases} \tag{61}$$

and

$$n = 2n_c \begin{cases} z_1^2: \begin{cases} \dot{h}_{2n_c,20}(\theta) - 2\mathrm{i}\omega_c h_{2n_c,20}(\theta) = (0,0)^T, \\ \dot{h}_{2n_c,20}(0) - \mathcal{L}_0\left(h_{2n_c,20}(\theta)\right) = \dfrac{1}{\sqrt{2\ell\pi}}\tilde{A}_{20}, \end{cases} \\ z_1 z_2: \begin{cases} \dot{h}_{2n_c,11}(\theta) = (0,0)^T, \\ \dot{h}_{2n_c,11}(0) - \mathcal{L}_0\left(h_{2n_c,11}(\theta)\right) = \dfrac{1}{\sqrt{2\ell\pi}}\tilde{A}_{11}. \end{cases} \end{cases} \tag{62}$$

Next, by combining with Equations (61) and (62), we will give the mathematical expressions of $h_{0,20}(\theta), h_{0,11}(\theta)$, $h_{2n_c,20}(\theta)$ and $h_{2n_c,11}(\theta)$.

Calculations of $\boldsymbol{h_{0,20}(\theta)}$ and $\boldsymbol{h_{0,11}(\theta)}$:

Notice that

$$\begin{cases} \dot{h}_{0,20}(\theta) - 2\mathrm{i}\omega_c h_{0,20}(\theta) = (0,0)^T, \\ \dot{h}_{0,20}(0) - L_0\left(h_{0,20}(\theta)\right) = \dfrac{1}{\sqrt{\ell\pi}}A_{20}, \end{cases} \tag{63}$$

then from (63), we have

$$h_{0,20}(\theta) = \mathrm{e}^{2\mathrm{i}\omega_c\theta} h_{0,20}(0)$$

and

$$\dot{h}_{0,20}(0) - 2\mathrm{i}\omega_c h_{0,20}(0) = (0,0)^T.$$

Notice that

$$L_0\big(h_{0,20}(\theta)\big) = \tau_c A h_{0,20}(0),$$

then we have

$$(2\mathrm{i}\omega_c I_2 - \tau_c A) h_{0,20}(0) = \frac{1}{\sqrt{\ell\pi}}A_{20},$$

and hence

$$h_{0,20}(\theta) = \mathrm{e}^{2\mathrm{i}\omega_c\theta} C_1$$

with

$$C_1 = (2\mathrm{i}\omega_c I_2 - \tau_c A)^{-1}\frac{1}{\sqrt{\ell\pi}}A_{20}.$$

Notice that

$$\begin{cases} \dot{h}_{0,11}(\theta) = (0,0)^T, \\ \dot{h}_{0,11}(0) - L_0\left(h_{0,11}(\theta)\right) = \dfrac{1}{\sqrt{\ell\pi}}A_{11}, \end{cases} \tag{64}$$

then from Equation (64), we have $h_{0,11}(\theta) = h_{0,11}(0)$ and $\dot{h}_{0,11}(0) = (0,0)^T$. Notice that

$$L_0\big(h_{0,11}(\theta)\big) = \tau_c A h_{0,11}(0),$$

then we have

$$-\tau_c A h_{0,11}(0) = \frac{1}{\sqrt{\ell\pi}}A_{11},$$

and hence $h_{0,11}(\theta) = C_2$ with

$$C_2 = (-\tau_c A)^{-1}\frac{1}{\sqrt{\ell\pi}}A_{11}.$$

Calculations of $\boldsymbol{h_{2n_c,20}(\theta)}$ and $\boldsymbol{h_{2n_c,11}(\theta)}$:

Notice that

$$\begin{cases} \dot{h}_{2n_c,20}(\theta) - 2\mathrm{i}\omega_c h_{2n_c,20}(\theta) = (0,0)^T, \\ \dot{h}_{2n_c,20}(0) - \mathcal{L}_0\left(h_{2n_c,20}(\theta)\right) = \dfrac{1}{\sqrt{2\ell\pi}}\tilde{A}_{20}, \end{cases} \tag{65}$$

then from Equation (65), we have $h_{2n_c,20}(\theta)=\mathrm{e}^{2\mathrm{i}\omega_c\theta}h_{2n_c,20}(0)$, and hence $h_{2n_c,20}(-1)=\mathrm{e}^{-2\mathrm{i}\omega_c}h_{2n_c,20}(0)$. Furthermore, from Equation (65) and

$$\mathcal{L}_0\big(h_{2n_c,20}(\theta)\big)=-\tau_c\frac{4n_c^2}{\ell^2}\Big(D_1h_{2n_c,20}(0)+D_2h_{2n_c,20}(-1)\Big)+\tau_cAh_{2n_c,20}(0),$$

we have

$$2\mathrm{i}\omega_ch_{2n_c,20}(0)=\frac{1}{\sqrt{2\ell\pi}}\tilde{A}_{20}-\tau_c\frac{4n_c^2}{\ell^2}\Big(D_1h_{2n_c,20}(0)+D_2h_{2n_c,20}(-1)\Big)+\tau_cAh_{2n_c,20}(0). \tag{66}$$

Therefore, by combining with $h_{2n_c,20}(-1)=\mathrm{e}^{-2\mathrm{i}\omega_c}h_{2n_c,20}(0)$ and (66), we can obtain

$$\left(2\mathrm{i}\omega_cI_2+\tau_c\frac{4n_c^2}{\ell^2}D_1+\tau_c\frac{4n_c^2}{\ell^2}D_2\mathrm{e}^{-2\mathrm{i}\omega_c}-\tau_cA\right)h_{2n_c,20}(0)=\frac{1}{\sqrt{2\ell\pi}}\tilde{A}_{20},$$

and hence

$$h_{2n_c,20}(\theta)=\mathrm{e}^{2\mathrm{i}\omega_c\theta}C_3$$

with

$$C_3=\left(2\mathrm{i}\omega_cI_2+\tau_c\frac{4n_c^2}{\ell^2}D_1+\tau_c\frac{4n_c^2}{\ell^2}D_2\mathrm{e}^{-2\mathrm{i}\omega_c}-\tau_cA\right)^{-1}\frac{1}{\sqrt{2\ell\pi}}\tilde{A}_{20}.$$

Here, $\tilde{A}_{20}$ and A_{20}^d are defined by Equations (60) and (49), respectively.

Notice that

$$\begin{cases}\dot{h}_{2n_c,11}(\theta)=(0,0)^T,\\ \dot{h}_{2n_c,11}(0)-\mathcal{L}_0\left(h_{2n_c,11}(\theta)\right)=\dfrac{1}{\sqrt{2\ell\pi}}\tilde{A}_{11},\end{cases} \tag{67}$$

then from Equation (67), we have $h_{2n_c,11}(\theta)=h_{2n_c,11}(0)$, and hence $h_{2n_c,11}(-1)=h_{2n_c,11}(0)$. Furthermore, by combining with Equation (67) and

$$\mathcal{L}_0\big(h_{2n_c,11}(\theta)\big)=-\tau_c\frac{4n_c^2}{\ell^2}\Big(D_1h_{2n_c,11}(0)+D_2h_{2n_c,11}(-1)\Big)+\tau_cAh_{2n_c,11}(0),$$

we have

$$(0,0)^T=-\tau_c\frac{4n_c^2}{\ell^2}\Big(D_1h_{2n_c,11}(0)+D_2h_{2n_c,11}(-1)\Big)+\tau_cAh_{2n_c,11}(0)+\frac{1}{\sqrt{2\ell\pi}}\tilde{A}_{11}. \tag{68}$$

Therefore, by combining with $h_{2n_c,11}(-1)=h_{2n_c,11}(0)$ and Equation (68), we can obtain

$$\left(\tau_c\frac{4n_c^2}{\ell^2}D_1+\tau_c\frac{4n_c^2}{\ell^2}D_2-\tau_cA\right)h_{2n_c,11}(0)=\frac{1}{\sqrt{2\ell\pi}}\tilde{A}_{11},$$

and hence

$$h_{2n_c,11}(\theta)=C_4$$

with

$$C_4=\left(\tau_c\frac{4n_c^2}{\ell^2}D_1+\tau_c\frac{4n_c^2}{\ell^2}D_2-\tau_cA\right)^{-1}\frac{1}{\sqrt{2\ell\pi}}\tilde{A}_{11}.$$

Here, $\tilde{A}_{11}$ and A_{11}^d are defined by Equations (60) and (49), respectively.

3.2. Calculations of $A_{q_1q_2}$ and $S_2(\Phi(\theta)z_x,w)$

In this subsection, let

$$F(\varphi,\mu)=\left(F^{(1)}(\varphi,\mu),F^{(2)}(\varphi,\mu)\right)^T$$

and $\varphi=(\varphi_1,\varphi_2)^T\in\mathcal{C}$, and we write

$$\frac{1}{j!}F_j(\varphi,0)=\sum_{j_1+j_2=j}\frac{1}{j_1!\,j_2!}f_{j_1j_2}\varphi_1^{j_1}(0)\varphi_2^{j_2}(0), \tag{69}$$

where

$$f_{j_1j_2} = \text{col}\left(f_{j_1j_2}^{(1)}, f_{j_1j_2}^{(2)}\right)$$

with

$$f_{j_1j_2}^{(k)} = \frac{\partial^{j_1+j_2}F^{(k)}(0,0)}{\partial\varphi_1^{j_1}(0)\,\partial\varphi_2^{j_2}(0)}, k = 1,2.$$

Then from Equation (69), we have

$$\begin{aligned} F_2(\varphi,\mu) &= F_2(\varphi,0) = 2\sum_{j_1+j_2=2}\frac{1}{j_1!\,j_2!}f_{j_1j_2}\varphi_1^{j_1}(0)\varphi_2^{j_2}(0) \\ &= f_{20}\varphi_1^2(0) + f_{02}\varphi_2^2(0) + 2f_{11}\varphi_1(0)\varphi_2(0) \end{aligned} \tag{70}$$

and

$$\begin{aligned} F_3(\varphi,0) &= 6\sum_{j_1+j_2=3}\frac{1}{j_1!\,j_2!}f_{j_1j_2}\varphi_1^{j_1}(0)\varphi_2^{j_2}(0) \\ &= f_{30}\varphi_1^3(0) + f_{03}\varphi_2^3(0) + 3f_{21}\varphi_1^2(0)\varphi_2(0) + 3f_{12}\varphi_1(0)\varphi_2^2(0). \end{aligned} \tag{71}$$

Notice that

$$\begin{aligned} \varphi(\theta) &= \Phi(\theta)z_x = \phi(\theta)z_1\gamma_{n_c}(x) + \bar{\phi}(\theta)z_2\gamma_{n_c}(x) \\ &= \begin{pmatrix} \phi_1(\theta)z_1\gamma_{n_c}(x) + \bar{\phi}_1(\theta)z_2\gamma_{n_c}(x) \\ \phi_2(\theta)z_1\gamma_{n_c}(x) + \bar{\phi}_2(\theta)z_2\gamma_{n_c}(x) \end{pmatrix} \\ &= \begin{pmatrix} \varphi_1(\theta) \\ \varphi_2(\theta) \end{pmatrix}, \end{aligned} \tag{72}$$

and similar to Equation (44), we have

$$F_2(\Phi(\theta)z_x,0) = \sum_{q_1+q_2=2} A_{q_1q_2}\gamma_{n_c}^{q_1+q_2}(x)z_1^{q_1}z_2^{q_2}, \tag{73}$$

then by combining with Equations (70), (72) and (73), we have

$$\begin{aligned} A_{20} &= f_{20}\phi_1^2(0) + f_{02}\phi_2^2(0) + 2f_{11}\phi_1(0)\phi_2(0), \\ A_{02} &= f_{20}\bar{\phi}_1^2(0) + f_{02}\bar{\phi}_2^2(0) + 2f_{11}\bar{\phi}_1(0)\bar{\phi}_2(0), \\ A_{11} &= 2f_{20}\phi_1(0)\bar{\phi}_1(0) + 2f_{02}\phi_2(0)\bar{\phi}_2(0) + 2f_{11}\left(\phi_1(0)\bar{\phi}_2(0) + \bar{\phi}_1(0)\phi_2(0)\right). \end{aligned}$$

Furthermore, by combining with Equations (44), (71) and (72), we have

$$\begin{aligned} A_{30} &= f_{30}\phi_1^3(0) + f_{03}\phi_2^3(0) + 3f_{21}\phi_1^2(0)\phi_2(0) + 3f_{12}\phi_1(0)\phi_2^2(0), \\ A_{03} &= f_{30}\bar{\phi}_1^3(0) + f_{03}\bar{\phi}_2^3(0) + 3f_{21}\bar{\phi}_1^2(0)\bar{\phi}_2(0) + 3f_{12}\bar{\phi}_1(0)\bar{\phi}_2^2(0), \\ A_{21} &= 3f_{30}\phi_1^2(0)\bar{\phi}_1(0) + 3f_{03}\phi_2^2(0)\bar{\phi}_2(0) + 3f_{21}(\phi_1^2(0)\bar{\phi}_2(0) + 2\phi_1(0)\bar{\phi}_1(0)\phi_2(0)) \\ &\quad +3f_{12}(2\phi_1(0)\phi_2(0)\bar{\phi}_2(0) + \bar{\phi}_1(0)\phi_2^2(0)), \\ A_{12} &= 3f_{30}\phi_1(0)\bar{\phi}_1^2(0) + 3f_{03}\phi_2(0)\bar{\phi}_2^2(0) + 3f_{21}(2\phi_1(0)\bar{\phi}_1(0)\bar{\phi}_2(0) + \bar{\phi}_1^2(0)\phi_2(0)) \\ &\quad +3f_{12}(\phi_1(0)\bar{\phi}_2^2(0) + 2\bar{\phi}_1(0)\phi_2(0)\bar{\phi}_2(0)). \end{aligned}$$

Moreover, from Equation (69), we have

$$\begin{aligned} F_2(\varphi(\theta)+w,\mu) &= F_2(\varphi(\theta)+w,0) = 2\sum_{j_1+j_2=2}\frac{1}{j_1!\,j_2!}f_{j_1j_2}(\varphi_1(0)+w_1(0))^{j_1}(\varphi_2(0)+w_2(0))^{j_2} \\ &= f_{20}(\varphi_1(0)+w_1(0))^2 + f_{02}(\varphi_2(0)+w_2(0))^2 + 2f_{11}(\varphi_1(0)+w_1(0))(\varphi_2(0)+w_2(0)). \end{aligned} \tag{74}$$

Notice that

$$\begin{aligned}\varphi(\theta)+w &= \Phi(\theta)z_x+w=\phi(\theta)z_1\gamma_{n_c}(x)+\bar{\phi}(\theta)z_2\gamma_{n_c}(x)+w\\ &=\begin{pmatrix}\phi_1(\theta)z_1\gamma_{n_c}(x)+\bar{\phi}_1(\theta)z_2\gamma_{n_c}(x)+w_1\\ \phi_2(\theta)z_1\gamma_{n_c}(x)+\bar{\phi}_2(\theta)z_2\gamma_{n_c}(x)+w_2\end{pmatrix}\\ &=\begin{pmatrix}\varphi_1(\theta)+w_1\\ \varphi_2(\theta)+w_2\end{pmatrix}\end{aligned}\tag{75}$$

and

$$\begin{aligned}F_2(\Phi(\theta)z_x+w,\mu) &= F_2(\Phi(\theta)z_x+w,0)\\ &=\sum_{q_1+q_2=2} A_{q_1q_2}\gamma_{n_c}^{q_1+q_2}(x)z_1^{q_1}z_2^{q_2}+S_2(\Phi(\theta)z_x,w)+O(|w|^2),\end{aligned}\tag{76}$$

then by combining with Equations (74)–(76), we have

$$\begin{aligned}&S_2(\Phi(\theta)z_x,w)\\ &=2f_{20}\big(\phi_1(0)z_1\gamma_{n_c}(x)+\bar{\phi}_1(0)z_2\gamma_{n_c}(x)\big)w_1(0)\\ &+2f_{02}\big(\phi_2(0)z_1\gamma_{n_c}(x)+\bar{\phi}_2(0)z_2\gamma_{n_c}(x)\big)w_2(0)\\ &+2f_{11}\Big(\big(\phi_1(0)z_1\gamma_{n_c}(x)+\bar{\phi}_1(0)z_2\gamma_{n_c}(x)\big)w_2(0)+\big(\phi_2(0)z_1\gamma_{n_c}(x)+\bar{\phi}_2(0)z_2\gamma_{n_c}(x)\big)w_1(0)\Big).\end{aligned}$$

4. Application to Holling-Tanner model with spatial memory and predator-taxis

In this section, we apply our newly developed algorithm in Section 2 to the Holling-Tanner model with spatial memory and predator-taxis, i.e., for the system (3), we let

$$\begin{aligned}f\big(u(x,t),v(x,t)\big)&=u(x,t)\big(1-\beta u(x,t)\big)-\frac{mu(x,t)v(x,t)}{1+u(x,t)},\\ g\big(u(x,t),v(x,t)\big)&=sv(x,t)\left(1-\frac{v(x,t)}{u(x,t)}\right),\end{aligned}\tag{77}$$

where $\beta>0, m>0$ and $s>0$. Thus, the system (3) becomes

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=d_{11}u_{xx}(x,t)+\xi\big(u(x,t)v_x(x,t)\big)_x+u(x,t)\big(1-\beta u(x,t)\big)-\dfrac{mu(x,t)v(x,t)}{1+u(x,t)}, x\in(0,\ell\pi), t>0,\\ \dfrac{\partial v(x,t)}{\partial t}=d_{22}v_{xx}(x,t)-d_{21}\big(v(x,t)u_x(x,t-\tau)\big)_x+sv(x,t)\left(1-\dfrac{v(x,t)}{u(x,t)}\right), x\in(0,\ell\pi), t>0,\\ u_x(0,t)=u_x(\ell\pi,t)=v_x(0,t)=v_x(\ell\pi,t)=0, t>0.\end{cases}\tag{78}$$

The Holling-tanner model is one of the typical predator-prey models. For the ordinary differential equation (78) with $d_{11}=\xi=d_{21}=d_{22}=0$, it has been completely analyzed in study[32]. For the diffusive model (78) with $\xi=d_{21}=0$, the global stability of the positive constant steady state was proved in studies[33,34], and the Hopf bifurcation and Turing instability have been studied in study[35].

4.1. Stability and Hopf bifurcation analysis

The system (78) has the positive constant steady state $E_*(u_*,v_*)$, where

$$u_*=v_*=\frac{1}{2\beta}\left(\sqrt{R^2+4\beta}-R\right)\tag{79}$$

with $R=\beta+m-1$. By combining with $E_*(u_*,v_*)$, (7) and (77), we have

$$\begin{aligned}&a_{11}=1-2\beta u_*-\frac{mu_*}{(1+u_*)^2}, a_{12}=-\frac{mu_*}{1+u_*}<0,\\ &a_{21}=s>0, a_{22}=-s<0.\end{aligned}\tag{80}$$

Moreover, by combining with Equations (6), (8), (9) and (80), the characteristic equation of system (78) can be written as

$$\Gamma_n(\lambda) = \det\left(\mathcal{M}_n(\lambda)\right) = \lambda^2 - T_n\lambda + \tilde{J}_n(\tau) = 0, \tag{81}$$

where

$$\begin{aligned} T_n &= \operatorname{Tr}(A) - \operatorname{Tr}(D_1)\frac{n^2}{\ell^2}, \\ \tilde{J}_n(\tau) &= \left(d_{11}d_{22} + d_{21}\xi u_* v_* \mathrm{e}^{-\lambda\tau}\right)\frac{n^4}{\ell^4} - \left(d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* + d_{21}v_* a_{12}\mathrm{e}^{-\lambda\tau}\right)\frac{n^2}{\ell^2} + \operatorname{Det}(A)\,. \end{aligned} \tag{82}$$

Notice that the mathematical expression in Equation (82) the same as in Equation (10). Furthermore, when $\tau = 0$, the characteristic Equation (81) becomes

$$\lambda^2 - T_n\lambda + \tilde{J}_n(0) = 0, \tag{83}$$

where

$$\tilde{J}_n(0) = (d_{11}d_{22} + d_{21}\xi u_* v_*)\frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* + d_{21}v_* a_{12})\frac{n^2}{\ell^2} + \operatorname{Det}(A). \tag{84}$$

A set of sufficient and necessary condition that all roots of Equation (83) have negative real parts is $T_n < 0, \tilde{J}_n(0) > 0$, which is always holds provided that $a_{11} < 0$, i.e.,

$$(C_0): 1 - 2\beta u_* - \frac{m u_*}{(1 + u_*)^2} < 0.$$

This implies that when $\tau = 0$ and the condition (C_0) holds, the positive constant steady state $E_*(u_*, v_*)$ is asymptotically stable for $d_{11} \geq 0, \xi \geq 0, d_{21} \geq 0$ and $d_{22} \geq 0$. Meanwhile, if we let $d_{21} = 0$, then by Equation (84), we denote

$$J_n := d_{11}d_{22}\frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_*)\frac{n^2}{\ell^2} + \operatorname{Det}(A).$$

It is easy to verify that $T_n < 0$ and $J_n > 0$ provided that the condition (C_0) holds. This implies that when $\tau = 0, d_{21} = 0$ and the condition (C_0) holds, the positive constant steady state $E_*(u_*, v_*)$ is asymptotically stable for $d_{11} \geq 0, \xi \geq 0$ and $d_{22} \geq 0$. Furthermore, since $\Gamma_n(0) = \tilde{J}_n(0) > 0$ under the condition (C_0), this implies that $\lambda = 0$ is not a root of Equation (81). Furthermore, let $\lambda = \mathrm{i}\omega_n (\omega_n > 0)$ be a root of (81). By substituting it along with expressions in Equation (82) into Equation (81), and separating the real part from the imaginary part, we have

$$\begin{cases} \omega_n^2 - J_n = \left(d_{21}\xi u_* v_* \dfrac{n^4}{\ell^4} - d_{21}v_* a_{12}\dfrac{n^2}{\ell^2}\right)\cos(\omega_n\tau), \\ -T_n\omega_n = \left(d_{21}\xi u_* v_* \dfrac{n^4}{\ell^4} - d_{21}v_* a_{12}\dfrac{n^2}{\ell^2}\right)\sin(\omega_n\tau), \end{cases} \tag{85}$$

which yields

$$\omega_n^4 + P_n\omega_n^2 + Q_n = 0, \tag{86}$$

where

$$P_n = T_n^2 - 2J_n = (d_{11}^2 + d_{22}^2)\frac{n^4}{\ell^4} - 2(d_{11}a_{11} + d_{22}a_{22} + a_{21}\xi u_*)\frac{n^2}{\ell^2} + a_{11}^2 + a_{22}^2 + 2a_{12}a_{21},$$

and

$$Q_n = \left(J_n + \left(d_{21}\xi u_* v_* \frac{n^4}{\ell^4} - d_{21}v_* a_{12}\frac{n^2}{\ell^2}\right)\right)\left(J_n - \left(d_{21}\xi u_* v_* \frac{n^4}{\ell^4} - d_{21}v_* a_{12}\frac{n^2}{\ell^2}\right)\right). \tag{87}$$

Notice that for Equation (86), it is easy to see that if

$$\text{either } P_n > 0 \text{ and } Q_n > 0 \text{ or } P_n^2 - 4Q_n < 0,$$

then Equation (86) has no positive root. Suppose that

$$Q_n > 0, P_n < 0 \text{ and } P_n^2 - 4Q_n > 0,$$

then Equation (86) has two positive roots. In addition, if

$$\text{either } Q_n < 0 \text{ or } Q_n = 0, P_n < 0 \text{ or } P_n < 0 \text{ and } P_n^2 - 4Q_n = 0,$$

then Equation (86) has only one positive root.

Case 4.1. It is easy to see that if the conditions (C_0) and

$$(C_1): P_n > 0 \text{ and } Q_n > 0 \text{ or } P_n^2 - 4Q_n < 0$$

hold, then Equation (86) has no positive roots. Hence, by combining with the Assumption 1, we know that all roots of Equation (81) have negative real parts when $\tau \in [0, +\infty)$ under the conditions (C_0) and (C_1).

If we set the parameters as follows

$$\ell = 2, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8, n = 2,$$

then by using the MATLAB software for auxiliary calculation, we can check that the conditions (C_0) and (C_1) are satisfied. Then the function images of $f(\omega_2) = \omega_2^4 + P_2\omega_2^2 + Q_2$ and $f(\omega_2) = 0$ are plotted in **Figure 1** which verifies the conclusion of Case 4.1.

In the following, we mainly consider the case of $Q_n < 0$, that is Equation (86) has only one positive root ω_n. In the following, we will discuss the case which is used to guarantee $Q_n < 0$ under the condition (C_0). When $\tau > 0$, according to Equations (81) and (87), we can define $Q_n = \Gamma_n(0)\tilde{Q}_n$ with

$$\Gamma_n(0) = \tilde{J}_n(0) = (d_{11}d_{22} + d_{21}\xi u_* v_*)\frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* + d_{21}v_* a_{12})\frac{n^2}{\ell^2} + \mathrm{Det}\,(A)$$

and

$$\tilde{Q}_n = (d_{11}d_{22} - d_{21}\xi u_* v_*)\frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_* a_{12})\frac{n^2}{\ell^2} + \mathrm{Det}(A), \quad (88)$$

and then by a simple analysis, we have $\Gamma_n(0) = \tilde{J}_n(0) > 0$ for any $n \in \mathbb{N}_0$. Therefore, the sign of Q_n coincides with that of $\tilde{Q}_n$, and in order to guaranteeing $Q_n < 0$, we only need to study the case of $\tilde{Q}_n < 0$.

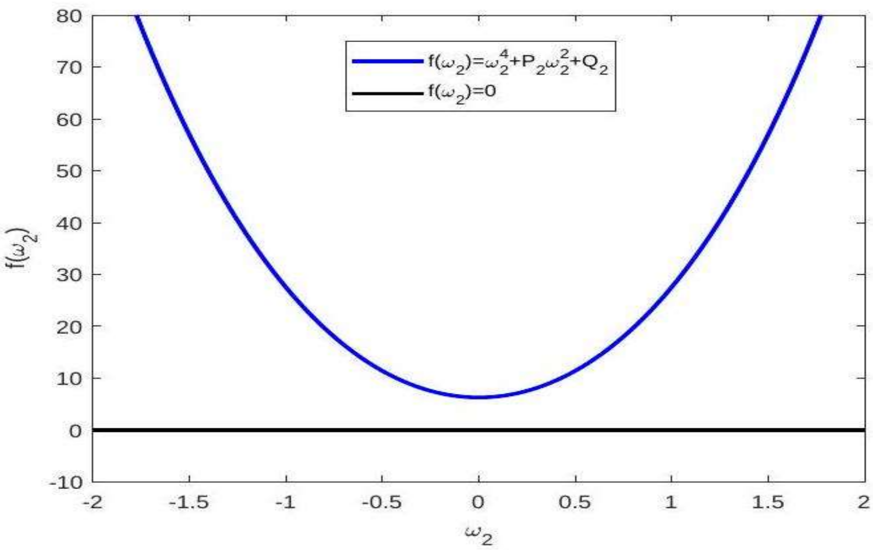


Figure 1. For the parameters $\ell = 2, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8, n = 2$, the function images of $f(\omega_2) = \omega_2^4 + P_2\omega_2^2 + Q_2$ and $f(\omega_2) = 0$.

Case 4.2. If $(d_{11}d_{22} - d_{21}\xi u_* v_*) > 0$ and the conditions (C_0),

$$(C_2): \mathrm{Det}\,(A) > 0, d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_* a_{12} > 0,$$
$$(d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_* a_{12})^2 - 4(d_{11}d_{22} - d_{21}\xi u_* v_*)\mathrm{Det}\,(A) > 0$$

hold, then from (88), $\tilde{Q}_n = 0$ has two positive roots. Without loss of generality, we assume that the two positive roots of $\tilde{Q}_n = 0$ are $\tilde{x}_1 = n_1^2/\ell^2$ and $\tilde{x}_2 = n_2^2/\ell^2$ with

$$\tilde{x}_{1,2} = \frac{\tilde{A}_1 \mp \sqrt{\tilde{A}_3}}{2\tilde{A}_2}, \quad (89)$$

where

$$\tilde{A}_1 = d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_*a_{12}, \tilde{A}_2 = d_{11}d_{22} - d_{21}\xi u_*v_*,$$
$$\tilde{A}_3 = \tilde{A}_1^2 - 4\tilde{A}_2\,\mathrm{Det}(A)\,. \tag{90}$$

Since $\tilde{x}_1 = n_1^2/\ell^2$ and $\tilde{x}_2 = n_2^2/\ell^2$, then $n_1 = \ell\sqrt{\tilde{x}_1}$ and $n_2 = \ell\sqrt{\tilde{x}_2}$. By using a geometric argument, we can conclude that

$$Q_n = \Gamma_n(0)\tilde{Q}_n \begin{cases} < 0, & n_1 < n < n_2, \\ \geq 0, & n \leq n_1 \text{ or } n \geq n_2, \end{cases}$$

where $n \in \mathbb{N}$. Therefore, Equation (86) has one positive root ω_n for $n_1 < n < n_2$ with $n \in \mathbb{N}$, where

$$\omega_n = \sqrt{\frac{-P_n + \sqrt{P_n^2 - 4Q_n}}{2}}. \tag{91}$$

Furthermore, by combining with the second mathematical expression in Equation (85), and by noticing that $a_{12} < 0, T_n < 0$ under the condition (C_0), then we have $\sin(\omega_n\tau) > 0$. Thus, by the first mathematical expression in Equation (85), we can set

$$\tau_{n,j} = \frac{1}{\omega_n}\left(\arccos\left(\frac{\omega_n^2 - J_n}{d_{21}\xi u_*v_*(n^4/\ell^4) - d_{21}v_*a_{12}(n^2/\ell^2)}\right) + 2j\pi\right), n \in \mathbb{N}, j \in \mathbb{N}_0. \tag{92}$$

Next, we continue to verify the transversality conditions for the Case 4.2.

Lemma 1. *Suppose that* $(d_{11}d_{22} - d_{21}\xi u_*v_*) > 0$*, the conditions* $(C_0), (C_2)$ *hold, and* $n_1 < n < n_2$ *with* $n \in \mathbb{N}$*, then we have*

$$\left.\frac{\mathrm{dRe}\,(\lambda(\tau))}{\mathrm{d}\tau}\right|_{\tau=\tau_{n,j}} > 0,$$

where Re $(\lambda(\tau))$ *represents the real part of* $\lambda(\tau)$.

Proof. By differentiating the two sides of

$$\Gamma_n(\lambda) = \det(\mathcal{M}_n(\lambda)) = \lambda^2 - T_n\lambda + \tilde{J}_n(\tau) = 0$$

with respect to τ, where T_n and $\tilde{J}_n(\tau)$ are defined by Equation (82), we have

$$\left(\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}\right)^{-1} = \frac{(2\lambda - T_n)\mathrm{e}^{\lambda\tau}}{-\lambda d_{21}v_*a_{12}(n^2/\ell^2) + \lambda d_{21}\xi u_*v_*(n^4/\ell^4)} - \frac{\tau}{\lambda}. \square \tag{93}$$

Therefore, by Equation (93), we have

$$\begin{aligned} \mathrm{Re}\left(\left.\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}\right|_{\tau=\tau_{n,j}}\right)^{-1} &= \mathrm{Re}\left(\frac{(2\mathrm{i}\omega_n - T_n)\mathrm{e}^{\mathrm{i}\omega_n\tau_{n,j}}}{-\mathrm{i}\omega_n d_{21}v_*a_{12}(n^2/\ell^2) + \mathrm{i}\omega_n d_{21}\xi u_*v_*(n^4/\ell^4)}\right) \\ &= \frac{2\cos(\omega_n\tau_{n,j})}{d_{21}\xi u_*v_*(n^4/\ell^4) - d_{21}v_*a_{12}(n^2/\ell^2)} - \frac{T_n\sin(\omega_n\tau_{n,j})}{\omega_n\left(d_{21}\xi u_*v_*(n^4/\ell^4) - d_{21}v_*a_{12}(n^2/\ell^2)\right)}. \end{aligned} \tag{94}$$

Furthermore, according to Equation (85), we have

$$\begin{aligned} \sin(\omega_n\tau_{n,j}) &= \frac{-T_n\omega_n}{d_{21}\xi u_*v_*(n^4/\ell^4) - d_{21}v_*a_{12}(n^2/\ell^2)}, \\ \cos(\omega_n\tau_{n,j}) &= \frac{\omega_n^2 - J_n}{d_{21}\xi u_*v_*(n^4/\ell^4) - d_{21}v_*a_{12}(n^2/\ell^2)}. \end{aligned} \tag{95}$$

Moreover, by combining with Equations (94), (95) and

$$\omega_n = \sqrt{\frac{-P_n + \sqrt{P_n^2 - 4Q_n}}{2}} > 0, Q_n < 0, a_{12} < 0,$$

we have

$$\begin{aligned}
\mathrm{Re}\left(\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=\tau_{n,j}}\right)^{-1} &= \frac{2\cos\left(\omega_n\tau_{n,j}\right)}{d_{21}\xi u_* v_*(n^4/\ell^4)-d_{21}v_* a_{12}(n^2/\ell^2)} - \frac{T_n \sin\left(\omega_n\tau_{n,j}\right)}{\omega_n\left(d_{21}\xi u_* v_*(n^4/\ell^4)-d_{21}v_* a_{12}(n^2/\ell^2)\right)} \\
&= \frac{2\omega_n^3+\omega_n(T_n^2-2J_n)}{\omega_n\left(d_{21}\xi u_* v_*(n^4/\ell^4)-d_{21}v_* a_{12}(n^2/\ell^2)\right)^2} \\
&= \frac{\sqrt{P_n^2-4Q_n}}{\left(d_{21}\xi u_* v_*(n^4/\ell^4)-d_{21}v_* a_{12}(n^2/\ell^2)\right)^2} > 0.
\end{aligned}$$

This, together with the fact that

$$\mathrm{sign}\left(\frac{\mathrm{dRe}\,(\lambda(\tau))}{\mathrm{d}\tau}\bigg|_{\tau=\tau_{n,j}}\right) = \mathrm{sign}\left(\mathrm{Re}\left(\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=\tau_{n,j}}\right)^{-1}\right)$$

completes the proof, where sign(.) represents the sign function.

Moreover, according to the above analysis, we have the following results.

Lemma 2. *Assume that the condition (C_0) is satisfied, then*

(i) if the condition (C_1) holds, then the positive constant steady state $E_*(u_*, v_*)$ of system (78) is locally asymptotically stable for all $\tau \geq 0$;

(ii) if $(d_{11}d_{22} - d_{21}\xi u_* v_*) > 0$, and the condition (C_2) holds, by denoting $\tau_* = min\{\tau_{n,0}: n_1 < n < n_2, n \in \mathbb{N}\}$, then the positive constant steady state $E_*(u_*, v_*)$ of system (78) is locally asymptotically stable for $0 \leq \tau < \tau_*$ and unstable for $\tau > \tau_*$. Furthermore, system (78) undergoes mode-n Hopf bifurcations at $\tau = \tau_{n,j}$ for $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$.

4.2. Numerical simulations

In this section, we verify the analytical results given in the previous sections by some numerical simulations and investigate the direction and stability of Hopf bifurcation. We use the following initial conditions for the system (78), i.e.,

$$u(x,t) = u_0(x), v(x,t) = v_0(x), t \in [-\tau, 0].$$

The normal form of Hopf bifurcation for system (78) can be calculated using our newly developed algorithm in Section 2, with detailed calculation procedures provided in the Appendix.

4.2.1. Mode-1 Hopf bifurcation

If we set the parameters as follows

$$\ell = 2, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8,$$

then we can easily obtain that

$$\begin{aligned}
&a_{11} = 1 - 2\beta u_* - \frac{mu_*}{(1+u_*)^2} = -0.5355 < 0, d_{11}d_{22} - d_{21}\xi u_* v_* = 3.84 > 0, \\
&\mathrm{Det}\,(A) = a_{11}a_{22} - a_{12}a_{21} = 0.6627 > 0, d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_* a_{12} = 4.1814 > 0, \\
&(d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_* a_{12})^2 - 4(d_{11}d_{22} - d_{21}\xi u_* v_*)\mathrm{Det}\,(A) = 7.3041 > 0.
\end{aligned}$$

Therefore, the conditions (C_0) and (C_2) are satisfied under the above parameter settings. In the following, we mainly verify the conclusion in Lemma 2 (ii). Furthermore, by combining with Equation (79) and (80), we have $E_*(u_*, v_*) = (1.4142, 1.4142)$,

$$a_{11} = -0.5355, a_{12} = -0.2929, a_{21} = 0.8, a_{22} = -0.8.$$

By combining with Equations (89)–(92), we have $n_1 = 0.8776, n_2 = 1.8935$, and consider that $n \in \mathbb{N}$, we have $\omega_{n_c} = \omega_1 = 0.418$ and $\tau_c = \tau_{1,0} = 6.1498$. Moreover, by Lemma 2 (ii), we have the following proposition.

Proposition 1. *For system (78) with the parameters $\ell = 2, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta =$*

$0.5, m = 0.5, s = 0.8$, *the positive constant steady state* $E_*(u_*, v_*)$ *of system (78) is asymptotically stable for* $0 \le \tau < \tau_{1,0} = 6.1498$ *and unstable for* $\tau > \tau_{1,0} = 6.1498$. *Furthermore, system (78) undergoes the mode-1 Hopf bifurcation at* $\tau = \tau_{1,0} = 6.1498$.

For the parameters $\ell = 2, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8$, according to Proposition 1, we know that system (78) undergoes a Hopf bifurcation at $\tau_{1,0} = 6.1498$. Furthermore, the direction and stability of Hopf bifurcation can be determined by calculating K_1K_2 and K_2 using the procedures developed in Section 2. After a direct calculation using MATLAB software, we obtain

$$K_1 = 0.016 > 0, K_2 = -0.9283 < 0, K_1K_2 = -0.0148 < 0,$$

which implies that the Hopf bifurcation at $\tau_{1,0} = 6.1498$ is supercritical and stable. When $\tau = 3 < \tau_{1,0} = 6.1498$, **Figure 2a,b** illustrates the evolution of the solution of system (78) starting from the initial values $u_0(x) = 1.4142 - 0.1\cos(x/2)$ and $v_0(x) = 1.4142 + 0.1\cos(x/2)$, finally converging to the positive constant steady state $E_*(u_*, v_*)$. Furthermore, when $\tau = 8 > \tau_{1,0} = 6.1498$, **Figure 3a–d** illustrates the existence of the spatially inhomogeneous periodic solution with the initial values $u_0(x) = 1.4142 - 0.1\cos(x/2)$ and $v_0(x) = 1.4142 + 0.1\cos(x/2)$.

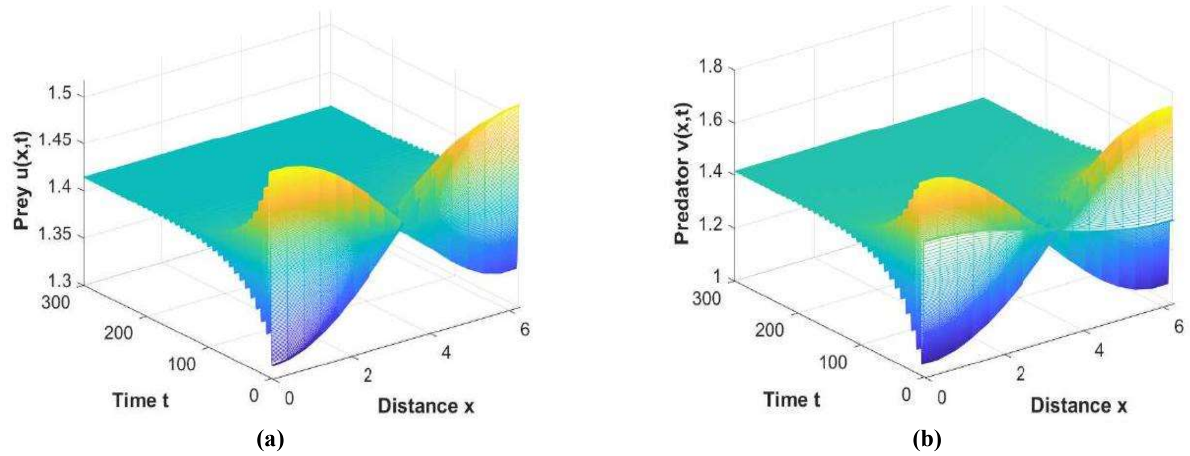


Figure 2. For the parameters $\ell = 2, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8$, when $\tau = 3 < \tau_{1,0} = 6.1498$, the positive constant steady state $E_*(u_*, v_*) = (1.4142, 1.4142)$ is locally asymptotically stable. The initial values are $u_0(x) = 1.4142 - 0.1\cos(x/2)$ and $v_0(x) = 1.4142 + 0.1\cos(x/2)$.

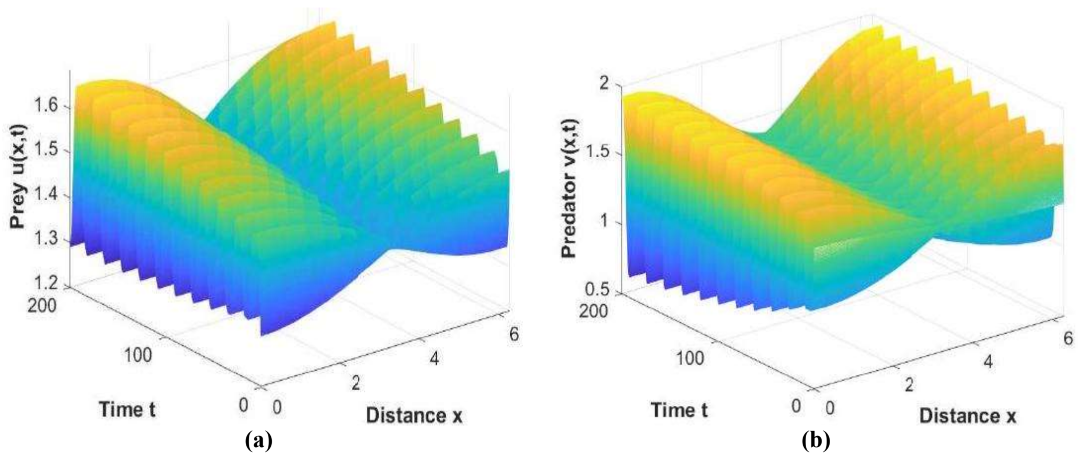
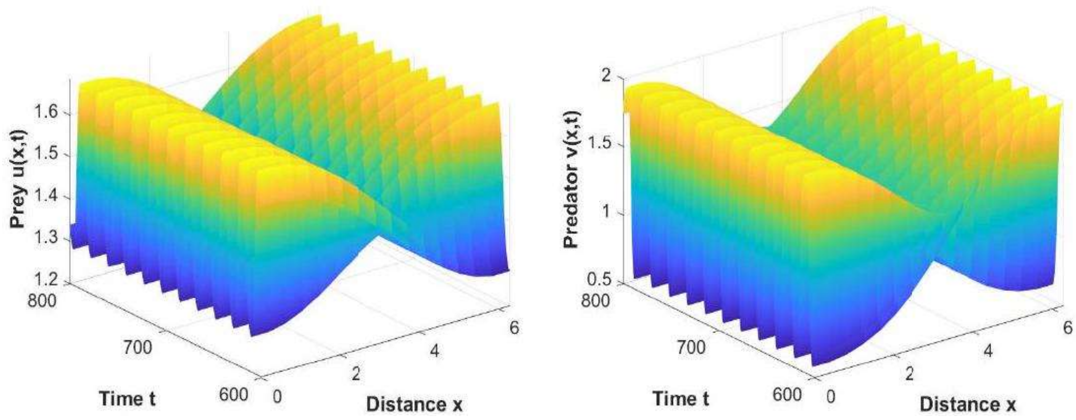


Figure 3. (*Continued*).



(c) (d)

Figure 3. For the parameters $\ell = 2, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8$, when $\tau = 8 > \tau_{1,0} = 6.1498$, there exists a stable spatially inhomogeneous periodic solution. **(a)** and **(b)** are the transient behaviours for $u(x,t)$ and $v(x,t)$, respectively, **(c)** and **(d)** are the long-term behaviours for $u(x,t)$ and $v(x,t)$, respectively. The initial values are $u_0(x) = 1.4142 - 0.1\cos(x/2)$ and $v_0(x) = 1.4142 + 0.1\cos(x/2)$.

4.2.2. Mode-2 Hopf bifurcation

If we set the parameters as follows

$$\ell = 3, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8,$$

then we can also easily obtain that

$$a_{11} = 1 - 2\beta u_* - \frac{mu_*}{(1+u_*)^2} = -0.5355 < 0, d_{11}d_{22} - d_{21}\xi u_* v_* = 3.84 > 0,$$

$$\text{Det}(A) = a_{11}a_{22} - a_{12}a_{21} = 0.6627 > 0, d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_*a_{12} = 4.1814 > 0,$$

$$(d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_*a_{12})^2 - 4(d_{11}d_{22} - d_{21}\xi u_* v_*)\text{Det}(A) = 7.3041 > 0.$$

Therefore, the conditions (C_0) and (C_2) are satisfied under the above parameter settings. In the following, we mainly verify the conclusion in Lemma 2 (ii). According to (79) and (80), we have $E_*(u_*, v_*) = (1.4142, 1.4142)$,

$$a_{11} = -0.5355, a_{12} = -0.2929, a_{21} = 0.8, a_{22} = -0.8.$$

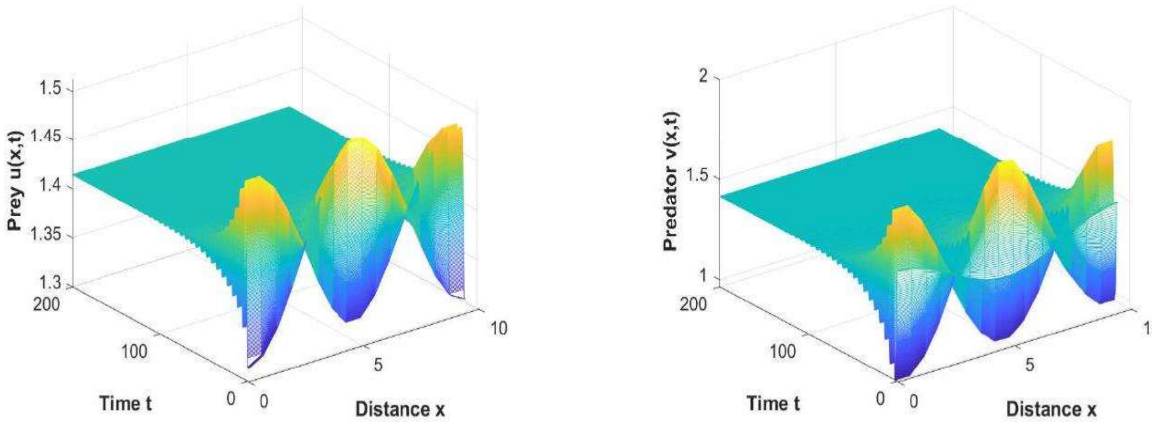
By combining with Equations (89)–(92), we have $n_1 = 1.3164, n_2 = 2.8403$, and consider that $n \in \mathbb{N}$, we have $\omega_{n_c} = \omega_2 = 0.6870$ and $\tau_c = \tau_{2,0} = 3.5361$. Moreover, by Lemma 2 (ii), we have the following proposition.

Proposition 2. *For system (78) with the parameters* $\ell = 3, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8$, *the positive constant steady state* $E_*(u_*, v_*)$ *of system (78) is asymptotically stable for* $0 \le \tau < \tau_{2,0} = 3.5361$ *and unstable for* $\tau > \tau_{2,0} = 3.5361$. *Furthermore, system (78) undergoes the mode-2 Hopf bifurcation at* $\tau = \tau_{2,0} = 3.5361$.

For the parameters $\ell = 3, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8$, according to Proposition 2, we know that system (78) undergoes a Hopf bifurcation at $\tau_{2,0} = 3.5361$. Furthermore, the direction and stability of Hopf bifurcation can be determined by calculating K_1K_2 and K_2 using the procedures developed in Section 2. After a direct calculation using MATLAB software, we obtain

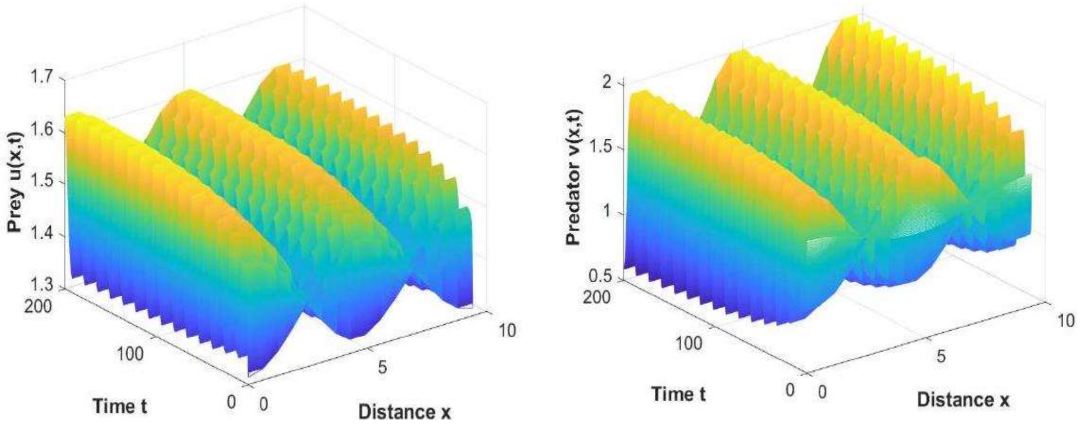
$$K_1 = 0.0410 > 0, K_2 = -1.3669 < 0, K_1K_2 = -0.0561 < 0,$$

which implies that the Hopf bifurcation at $\tau_{2,0} = 3.5361$ is supercritical and stable. When $\tau = 2 < \tau_{2,0} = 3.5361$, **Figure 4a** and **b** illustrate the evolution of the solution of system (78) starting from the initial values $u_0(x) = 1.4142 - 0.1\cos(2x/3)$ and $v_0(x) = 1.4142 + 0.1\cos(2x/3)$, finally converging to the positive constant steady state $E_*(u_*, v_*)$. Furthermore, when $\tau = 6 > \tau_{2,0} = 3.5361$, **Figure 5a–d** illustrate the existence of the spatially inhomogeneous periodic solution with the initial values $u_0(x) = 1.4142 - 0.1\cos(2x/3)$ and $v_0(x) = 1.4142 + 0.1\cos(2x/3)$.

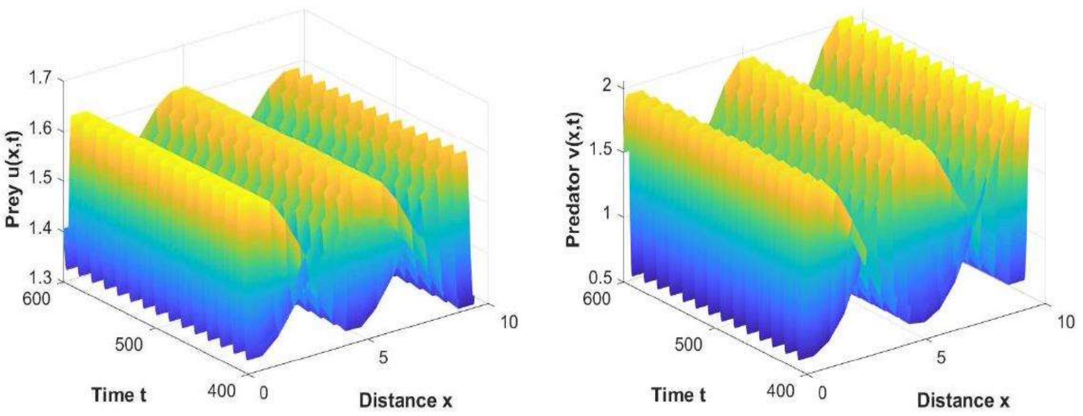


(a) **(b)**

Figure 4. For the parameters $\ell = 3, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8$, when $\tau = 2 < \tau_{2,0} = 3.5361$, the positive constant steady state $E_*(u_*, v_*) = (1.4142, 1.4142)$ is locally asymptotically stable. The initial values are $u_0(x) = 1.4142 - 0.1\cos(2x/3)$ and $v_0(x) = 1.4142 + 0.1\cos(2x/3)$.



(a) **(b)**



(c) **(d)**

Figure 5. For the parameters $\ell = 3, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8$, when $\tau = 6 > \tau_{2,0} = 3.5361$, there exists a stable spatially inhomogeneous periodic solution. **(a)** and **(b)** are the transient behaviours for $u(x,t)$ and $v(x,t)$, respectively, **(c)** and **(d)** are the long-term behaviours for $u(x,t)$ and $v(x,t)$, respectively. The initial values are $u_0(x) = 1.4142 - 0.1\cos(2x/3)$ and $v_0(x) = 1.4142 + 0.1\cos(2x/3)$.

5. Conclusion and discussion

In this paper, the diffusive predator-prey system with spatial memory and predator-taxis is proposed, and we derive an algorithm for calculating the normal form of Hopf bifurcation for this system. As a real application, we consider the Holling-Tanner model with spatial memory and predator-taxis. Then we study the dynamics of this system. Firstly, the inhomogeneous spatial patterns, i.e., two stable spatially inhomogeneous periodic solutions are found. Secondly, the supercritical and stable mode-1 and mode-2 Hopf bifurcation periodic solutions are found by using the newly developed algorithm. Furthermore, numerical simulations verify our theoretical analysis results and give us a more intuitive display.

It is worth mentioning that in this paper, the delay only occurs in the diffusion term, not in the reaction form for our proposed diffusive predator-prey system with spatial memory and predator-taxis. However, the gestation, hunting, migration and maturation delays, etc., often occur in the reaction term for the general predator-prey models. By noticing this point, on the basis of system (3), the system

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_{11}u_{xx}(x,t) + \xi\big(u(x,t)v_x(x,t)\big)_x + f\big(u(x,t),v(x,t),u(x,t-\sigma),v(x,t-\sigma)\big), \\ \qquad\qquad x \in (0,\ell\pi), t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = d_{22}v_{xx}(x,t) - d_{21}\big(v(x,t)u_x(x,t-\tau)\big)_x + g\big(u(x,t),v(x,t),u(x,t-\sigma),v(x,t-\sigma)\big), \\ \qquad\qquad x \in (0,\ell\pi), t > 0, \end{cases}$$

which needs further research, where $\sigma > 0$ is the delay occurs in the reaction term.

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Conflict of interest

The author declares no conflict of interest.

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Appendix

Remark 1. *Assume that at* $\tau = \tau_c$*, the characteristic Equation (81) has a pair of purely imaginary roots* $\pm i\omega_{n_c}$ *with* $\omega_{n_c} > 0$ *for* $n = n_c \in \mathbb{N}$ *and all other eigenvalues have negative real parts. Let* $\lambda(\tau) = \alpha_1(\tau) \pm i\alpha_2(\tau)$ *be a pair of roots of the characteristic Equation (81) near* $\tau = \tau_c$ *satisfying* $\alpha_1(\tau_c) = 0$ *and* $\alpha_2(\tau_c) = \omega_{n_c}$*. In addition, the corresponding transversality condition holds.*

The normal form of Hopf bifurcation for system (78) can be calculated by using our newly developed algorithm in Section 2. Here, we give the detail calculation procedures of $B_1, B_{21}, B_{22}, B_{23}$ steps by steps.

Step 1:

$$B_1 = 2\psi^T(0)\left(A\phi(0) - \frac{n_c^2}{\ell^2}(D_1\phi(0) + D_2\phi(-1))\right)$$

with

$$D_1 = \begin{pmatrix} d_{11} & \xi u_* \\ 0 & d_{22} \end{pmatrix}, D_2 = \begin{pmatrix} 0 & 0 \\ -d_{21}v_* & 0 \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Here,

$$\phi = \begin{pmatrix} 1 \\ \dfrac{a_{11} - \mathrm{i}\omega_{n_c} - d_{11}(n_c^2/\ell^2)}{\xi u_*(n_c^2/\ell^2) - a_{12}} \end{pmatrix}, \psi = \eta \begin{pmatrix} 1 \\ \dfrac{a_{12} - \xi u_*(n_c^2/\ell^2)}{\mathrm{i}\omega_{n_c} + d_{22}(n_c^2/\ell^2) - a_{22}} \end{pmatrix}$$

with

$$\eta = \frac{\mathrm{i}\omega_{n_c} + (n_c/\ell)^2 d_{22} - a_{22}}{2\mathrm{i}\omega_{n_c} + (n_c/\ell)^2 d_{11} - a_{11} + (n_c/\ell)^2 d_{22} - a_{22} + \tau_c a_{12} d_{21} v_* (n_c/\ell)^2 \mathrm{e}^{-\mathrm{i}\omega_c}}.$$

Step 2:

$$B_{21} = \frac{3}{2\ell\pi}\psi^T A_{21}$$

with

$$\begin{aligned} A_{21} &= 3f_{30}\phi_1^2(0)\bar{\phi}_1(0) + 3f_{03}\phi_2^2(0)\bar{\phi}_2(0) + 3f_{21}(\phi_1^2(0)\bar{\phi}_2(0) + 2\phi_1(0)\bar{\phi}_1(0)\phi_2(0)) \\ &\quad + 3f_{12}(2\phi_1(0)\phi_2(0)\bar{\phi}_2(0) + \bar{\phi}_1(0)\phi_2^2(0)). \end{aligned}$$

Here,

$$\begin{aligned} f_{30}^{(1)} &= -6\tau_c m(1+u_*)^{-4}v_*, f_{30}^{(2)} = 6\tau_c s u_*^{-4} v_*^2, \\ f_{21}^{(1)} &= 2\tau_c m(1+u_*)^{-3}, f_{21}^{(2)} = -4\tau_c s u_*^{-3} v_*, \\ f_{12}^{(1)} &= 0, f_{12}^{(2)} = 2\tau_c s u_*^{-2}, \\ f_{03}^{(1)} &= 0, f_{03}^{(2)} = 0. \end{aligned}$$

Step 3:

$$\begin{aligned} B_{22} &= \frac{1}{\sqrt{\ell\pi}}\psi^T\left(S_2(\phi(\theta), h_{0,11}(\theta)) + S_2(\bar{\phi}(\theta), h_{0,20}(\theta))\right) \\ &\quad + \frac{1}{\sqrt{2\ell\pi}}\psi^T\left(S_2(\phi(\theta), h_{2n_c,11}(\theta)) + S_2(\bar{\phi}(\theta), h_{2n_c,20}(\theta))\right) \end{aligned}$$

with

$$\begin{aligned}
S_2\big(\phi(\theta),h_{0,11}(\theta)\big) &= 2f_{20}\phi_1(0)h_{0,11}^{(1)}(0)+2f_{02}\phi_2(0)h_{0,11}^{(2)}(0)\\
&\quad +2f_{11}\left(\phi_1(0)h_{0,11}^{(2)}(0)+\phi_2(0)h_{0,11}^{(1)}(0)\right),\\
S_2\big(\bar{\phi}(\theta),h_{0,20}(\theta)\big) &= 2f_{20}\bar{\phi}_1(0)h_{0,20}^{(1)}(0)+2f_{02}\bar{\phi}_2(0)h_{0,20}^{(2)}(0)\\
&\quad +2f_{11}\left(\bar{\phi}_1(0)h_{0,20}^{(2)}(0)+\bar{\phi}_2(0)h_{0,20}^{(1)}(0)\right),\\
S_2\big(\phi(\theta),h_{2n_c,11}(\theta)\big) &= 2f_{20}\phi_1(0)h_{2n_c,11}^{(1)}(0)+2f_{02}\phi_2(0)h_{2n_c,11}^{(2)}(0)\\
&\quad +2f_{11}\left(\phi_1(0)h_{2n_c,11}^{(2)}(0)+\phi_2(0)h_{2n_c,11}^{(1)}(0)\right),\\
S_2\big(\bar{\phi}(\theta),h_{2n_c,20}(\theta)\big) &= 2f_{20}\bar{\phi}_1(0)h_{2n_c,20}^{(1)}(0)+2f_{02}\bar{\phi}_2(0)h_{2n_c,20}^{(2)}(0)\\
&\quad +2f_{11}\left(\bar{\phi}_1(0)h_{2n_c,20}^{(2)}(0)+\bar{\phi}_2(0)h_{2n_c,20}^{(1)}(0)\right).
\end{aligned}$$

Here,

$$\begin{aligned}
&f_{20}^{(1)}=-2\tau_c\beta+2\tau_c m(1+u_*)^{-3}v_*, f_{20}^{(2)}=-2\tau_c s u_*^{-3}v_*^2,\\
&f_{11}^{(1)}=-\tau_c m(1+u_*)^{-2}, f_{11}^{(2)}=2\tau_c s u_*^{-2}v_*,\\
&f_{02}^{(1)}=0, f_{02}^{(2)}=-2\tau_c s u_*^{-1}.
\end{aligned}$$

Furthermore, we have

$$\begin{cases}
h_{0,20}(\theta)=\dfrac{1}{\sqrt{\ell\pi}}\left(\widetilde{\mathcal{M}}_0(2\mathrm{i}\omega_c)\right)^{-1}A_{20}\mathrm{e}^{2\mathrm{i}\omega_c\theta},\\
h_{0,11}(\theta)=\dfrac{1}{\sqrt{\ell\pi}}\left(\widetilde{\mathcal{M}}_0(0)\right)^{-1}A_{11}
\end{cases}$$

and

$$\begin{cases}
h_{2n_c,20}(\theta)=\dfrac{1}{\sqrt{2\ell\pi}}\left(\widetilde{\mathcal{M}}_{2n_c}(2\mathrm{i}\omega_c)\right)^{-1}\tilde{A}_{20}\mathrm{e}^{2\mathrm{i}\omega_c\theta},\\
h_{2n_c,11}(\theta)=\dfrac{1}{\sqrt{2\ell\pi}}\left(\widetilde{\mathcal{M}}_{2n_c}(0)\right)^{-1}\tilde{A}_{11}
\end{cases}$$

with

$$\widetilde{\mathcal{M}}_n(\lambda)=\lambda I_2+\tau_c(n/\ell)^2D_1+\tau_c(n/\ell)^2\mathrm{e}^{-\lambda}D_2-\tau_c A.$$

Here,

$$\begin{aligned}
A_{20}&=f_{20}\phi_1^2(0)+f_{02}\phi_2^2(0)+2f_{11}\phi_1(0)\phi_2(0),\\
A_{11}&=2f_{20}\phi_1(0)\bar{\phi}_1(0)+2f_{02}\phi_2(0)\bar{\phi}_2(0)+2f_{11}(\phi_1(0)\bar{\phi}_2(0)+\bar{\phi}_1(0)\phi_2(0))
\end{aligned}$$

and

$$\begin{cases}
\tilde{A}_{20}=A_{20}-2(n_c/\ell)^2A_{20}^d,\\
\tilde{A}_{11}=A_{11}-2(n_c/\ell)^2A_{11}^d
\end{cases}$$

with

$$\begin{cases}
A_{20}^d=\begin{pmatrix}2\xi\tau_c\phi_1(0)\phi_2(0)\\-2d_{21}\tau_c\phi_1(-1)\phi_2(0)\end{pmatrix}=\bar{A}_{02}^d,\\
A_{11}^d=\begin{pmatrix}4\xi\tau_c\mathrm{Re}\,\{\phi_1(0)\bar{\phi}_2(0)\}\\-4d_{21}\tau_c\mathrm{Re}\,\{\phi_1(-1)\bar{\phi}_2(0)\}\end{pmatrix}.
\end{cases}$$

Step 4:

$$\begin{aligned}
B_{23} &= -\frac{1}{\sqrt{\ell\pi}}(n_c/\ell)^2\psi^T\left(S_2^{(d,1)}\big(\phi(\theta),h_{0,11}(\theta)\big)+S_2^{(d,1)}\big(\bar{\phi}(\theta),h_{0,20}(\theta)\big)\right)\\
&\quad +\frac{1}{\sqrt{2\ell\pi}}\psi^T\sum_{j=1,2,3}b_{2n_c}^{(j)}\left(S_2^{(d,j)}\big(\phi(\theta),h_{2n_c,11}(\theta)\big)+S_2^{(d,j)}\big(\bar{\phi}(\theta),h_{2n_c,20}(\theta)\big)\right)
\end{aligned}$$

with

$$b_{2n_c}^{(1)} = -\frac{n_c^2}{\ell^2}, b_{2n_c}^{(2)} = \frac{2n_c^2}{\ell^2}, b_{2n_c}^{(3)} = -\frac{(2n_c)^2}{\ell^2}$$

and

$$\begin{cases} S_2^{(d,1)}\left(\phi(\theta), h_{0,11}(\theta)\right) = 2\begin{pmatrix} \xi\tau_c\phi_2(0)h_{0,11}^{(1)}(0) \\ -d_{21}\tau_c\phi_1(-1)h_{0,11}^{(2)}(0)\end{pmatrix}, \\ S_2^{(d,1)}\left(\bar{\phi}(\theta), h_{0,20}(\theta)\right) = 2\begin{pmatrix} \xi\tau_c\bar{\phi}_2(0)h_{0,20}^{(1)}(0) \\ -d_{21}\tau_c\bar{\phi}_1(-1)h_{0,20}^{(2)}(0)\end{pmatrix}, \\ S_2^{(d,1)}\left(\phi(\theta), h_{2n_c,11}(\theta)\right) = 2\begin{pmatrix} \xi\tau_c\phi_2(0)h_{2n_c,11}^{(1)}(0) \\ -d_{21}\tau_c\phi_1(-1)h_{2n_c,11}^{(2)}(0)\end{pmatrix}, \\ S_2^{(d,2)}\left(\phi(\theta), h_{2n_c,11}(\theta)\right) = 2\begin{pmatrix} \xi\tau_c\left(\phi_2(0)h_{2n_c,11}^{(1)}(0) + \phi_1(0)h_{2n_c,11}^{(2)}(0)\right) \\ -d_{21}\tau_c\left(\phi_2(0)h_{2n_c,11}^{(1)}(-1) + \phi_1(-1)h_{2n_c,11}^{(2)}(0)\right)\end{pmatrix}, \\ S_2^{(d,3)}\left(\phi(\theta), h_{2n_c,11}(\theta)\right) = 2\begin{pmatrix} \xi\tau_c\phi_1(0)h_{2n_c,11}^{(2)}(0) \\ -d_{21}\tau_c\phi_2(0)h_{2n_c,11}^{(1)}(-1)\end{pmatrix}, \\ S_2^{(d,1)}\left(\bar{\phi}(\theta), h_{2n_c,20}(\theta)\right) = 2\begin{pmatrix} \xi\tau_c\bar{\phi}_2(0)h_{2n_c,20}^{(1)}(0) \\ -d_{21}\tau_c\bar{\phi}_1(-1)h_{2n_c,20}^{(2)}(0)\end{pmatrix}, \\ S_2^{(d,2)}\left(\bar{\phi}(\theta), h_{2n_c,20}(\theta)\right) = 2\begin{pmatrix} \xi\tau_c\left(\bar{\phi}_2(0)h_{2n_c,20}^{(1)}(0) + \bar{\phi}_1(0)h_{2n_c,20}^{(2)}(0)\right) \\ -d_{21}\tau_c\left(\bar{\phi}_2(0)h_{2n_c,20}^{(1)}(-1) + \bar{\phi}_1(-1)h_{2n_c,20}^{(2)}(0)\right)\end{pmatrix}, \\ S_2^{(d,3)}\left(\bar{\phi}(\theta), h_{2n_c,20}(\theta)\right) = 2\begin{pmatrix} \xi\tau_c\bar{\phi}_1(0)h_{2n_c,20}^{(2)}(0) \\ -d_{21}\tau_c\bar{\phi}_2(0)h_{2n_c,20}^{(1)}(-1)\end{pmatrix}. \end{cases}$$