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ORIGINAL RESEARCH ARTICLE

On types of stability in Hamiltonian systems

Alexander Dmitrievich Bruno[1,*], Alexander Borisovich Batkhin[2]

[1] *Keldysh Institute of Applied Mathematics of RAS, Moscow 125047, Russia*

[2] *Department of Aerospace Engineering, Technion—Israel Institute of Technology, Haifa 3200003, Israel*

*** Corresponding author:** Alexander Dmitrievich Bruno, abruno@keldysh.ru

ABSTRACT

We consider conditions of three types of stability: Lyapunov, formal and weak of a stationary solution, and of a periodic solution in a Hamiltonian system with a finite number of degrees of freedom. The conditions contain restrictions on the order of resonances and some inequalities for initial coefficients of the normal forms of the Hamiltonian functions. We show that the number-theoretical analysis of frequencies can help in proof of stability. We also estimate the orders of solutions' divergence from the stationary or the periodic ones under lack of formal stability.

Keywords: stationary solution; periodic solution; normal form; formal stability; weak stability

1. Introduction

Nowadays, there are three types of definitions of stationary-point stability in a Hamiltonian system: Lyapunov stability, formal stability by Moser, and formal stability by Markeev. In Section 2, we present these definitions for a stationary point and give conditions on the Hamiltonian function that guarantee them. It is shown that a theoretical-numerical analysis of frequencies can help in proof of stability. In the absence of formal stability, one can consider weak stability in the situation when the order of scattering of solutions is small. Therefore, the order of scattering of the solution from a stationary point in the absence of formal stability is estimated. In Section 3, the conditions for formal orbital stability according to Moser of the periodic solution of the Hamilton system are given and the proof of such stability is presented. We also present estimates of the order of divergence of solutions from the periodic one in the absence of formal orbital stability.

Note on notation

Vector magnitudes are indicated in bold type. By default, these are vectors of dimension n unless otherwise specified, i.e., $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{p} = (p_1, \dots, p_n)$, and $\mathbf{x}^{\mathbf{p}} = x_1^{p_1} \cdots x_n^{p_n}$; the scalar product $\langle \mathbf{p}, \mathbf{q} \rangle = p_1 q_1 + \cdots + p_n q_n$; $\| \mathbf{p} \| = |p_1| + \cdots + |p_n|$.

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2. Vicinity of a stationary point

2.1. Resonant normal form

Consider a Hamiltonian system

$$\dot{\xi}_j = \frac{\partial \gamma}{\partial \eta_j}, \qquad \dot{\eta}_j = -\frac{\partial \gamma}{\partial \xi_j}, \qquad j = 1, \dots, n \tag{1}$$

with n degrees of freedom in the neighborhood of a stationary point at the origin

$$\boldsymbol{\zeta} \overset{\text{def}}{=} (\boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \tag{2}$$

If the Hamilton function $\gamma(\boldsymbol{\zeta})$ is analytic at this point, then it expands into a convergent power series

$$\gamma(\boldsymbol{\zeta}) = \sum \gamma_{\mathbf{pq}} \boldsymbol{\xi}^{\mathbf{p}} \boldsymbol{\eta}^{\mathbf{q}}, \tag{3}$$

where $\mathbf{p}$, $\mathbf{q} \in \mathbb{Z}^n$, $\mathbf{p}, \mathbf{q} \geq 0$, $\gamma_{\mathbf{pq}}$ are constant coefficients. Since the point (2) is stationary, the expansion of (3) starts with quadratic terms. They correspond to the linear part of the system (1).

The eigenvalues of its matrix are divided into pairs $\lambda_{j+n} = -\lambda_j$, $j = 1, \dots, n$. Denote by vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$, the set of basic eigenvalues. As known, canonical coordinate substitutions

$$\boldsymbol{\xi}, \boldsymbol{\eta} \to \mathbf{x}, \mathbf{y} \tag{4}$$

preserve the Hamiltonian nature of the system.

Theorem 2.1. There is a canonical formal transformation (4) that reduces the Hamiltonian (3) to the normal form[1]

$$g(\mathbf{x}, \mathbf{y}) = \sum g_{\mathbf{pq}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}}, \tag{5}$$

where the series g contains only resonant terms with

$$\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\lambda} \rangle = 0.$$

If $\boldsymbol{\lambda} \neq 0$, then the normal form (5) is equivalent to a system with fewer degrees of freedom and additional parameters[2].

For the real initial system (1), the constant coefficients $g_{\mathbf{pq}}$ of the complex normal form (5) satisfy special relations, and the standard canonical linear coordinate substitution $\mathbf{x}, \mathbf{y} \to \mathbf{X}, \mathbf{Y}$ reduces the system (5) into a real system.

Definition 2.1. For each resonance are defined:

- multiplicity $\mathfrak{k}$: the number of linearly independent solutions $\mathbf{p} \in \mathbb{Z}^n$ to the resonant equation

$$\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0 \tag{6}$$

- order $\mathfrak{q}$: $\mathfrak{q} = min \parallel \mathbf{p} \parallel$ over $\mathbf{p} \in \mathbb{Z}^n \backslash \{0\}$, satisfying (6);
- n-frequency resonance: if exactly n nonzero eigenvalues λ_j are included in the nontrivial solution of the resonance equation;
- Strong resonances are called the resonances of orders 2, 3, or 4.

Condition A_k^n for system with n degrees of freedom takes place if the resonant Equation (6) has no integer solutions $\mathbf{p} \in \mathbb{Z}^n$ with $\parallel \mathbf{p} \parallel \leq k$.

This condition means that there are no resonances up to and including the order k. If it is satisfied, then in the normal form (5)

$$g = \sum_{l=1}^{[k/2]} g_l(\mathbf{r}) + \tilde{g}^{(k)}(\mathbf{x}, \mathbf{y}) \tag{7}$$

where $g_l(\mathbf{r})$ are homogeneous polynomials from $r_j = x_j y_j$, $j = 1, \dots, n$, of degree l, and $\tilde{g}^{(k)}$ is a series from $\mathbf{x}, \mathbf{y}$ starting with powers above k and $[k/2]$ means an integer part of number $k/2$.

Thus, it is possible to obtain a Hamiltonian of the form (7) with partial normalization only up to order k when $\tilde{g}^{(k)}$ contains not only resonance terms.

In particular, under the condition A_2^n we have

$$g = \langle \mathbf{r}, \boldsymbol{\lambda} \rangle + \tilde{g}^{(3)}(\mathbf{x}, \mathbf{y}) \tag{8}$$

and under the condition A_4^n we have

$$g = \langle \mathbf{r}, \boldsymbol{\lambda} \rangle + \langle C\mathbf{r}, \mathbf{r} \rangle + \tilde{g}^{(5)}(\mathbf{x}, \mathbf{y}) \tag{9}$$

where C is $n \times n$ matrix.

2.2. Stability

Definition 2.2. A stationary point $\boldsymbol{\zeta} = 0$ of a real Hamiltonian system (1) is *stable by Lyapunov* if for every $\varepsilon > 0$ in "cube" $\parallel \boldsymbol{\zeta} \parallel < \varepsilon$ there exists a closed integral $(2n-1)$-dimensional variety $\mathcal{L}$ surrounding the point $\boldsymbol{\zeta} = 0$ from all sides.

Lemma 2.1. A stationary point $\boldsymbol{\zeta} = 0$ is Lyapunov stable if there exists a sign-definite real integral

$$f(\boldsymbol{\zeta}) = f_l(\boldsymbol{\zeta}) + \tilde{f}^{(l)}(\boldsymbol{\zeta}) \tag{10}$$

of the system (1), where $f_l(\boldsymbol{\zeta})$ is a homogeneous form of degree l. In other words,

$$\sum_{j=1} \left(\frac{\partial f}{\partial x_j} \frac{\partial \gamma}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial \gamma}{\partial x_j} \right) = 0 \tag{11}$$

and $f_l(\boldsymbol{\zeta})$ does not equal to zero at any $\boldsymbol{\zeta}$ except the point $\boldsymbol{\zeta} = 0$.

Stability is possible only if $Re\boldsymbol{\lambda} = 0$. If the condition A_2^n is satisfied, then all λ_j are different and non-zero. In this case, the complex coordinates $\mathbf{x}, \mathbf{y}$ are related to the real coordinates. $\mathbf{X}, \mathbf{Y}$ by the canonical substitution

$$x_j = \frac{iX_j - Y_j}{\sqrt{2i}}, \qquad y_j = \frac{iX_j + Y_j}{\sqrt{2i}}, \qquad j = 1, \dots, n \tag{12}$$

With complex conjugation

$$\bar{x}_j = -iy_j, \qquad \bar{y}_j = -ix_j, \qquad j = 1, \dots, n \tag{13}$$

the Hamiltonian function $g(\mathbf{x}, \mathbf{y})$ goes into itself, that is, into (5):

$$g_{\mathbf{pq}} = \bar{g}_{\mathbf{qp}} (-i)^{\|\mathbf{p}+\mathbf{q}\|} \tag{14}$$

as far as $p_j, q_j \geq 0$. Suppose

$$X_j^2 + Y_j^2 = R_j, \qquad \lambda_j = i\alpha_j, \qquad j = 1, \dots, n \tag{15}$$

Then in real coordinates, $R_j \geq 0$ and α_j is real,

$$r_j = x_j y_j = \frac{i}{2}\left(X_j^2 + Y_j^2\right) = \frac{i}{2} R_j, \qquad j = 1, \dots, n \tag{16}$$

$$\sum_{j=1}^{n} \lambda_j r_j = -\frac{1}{2} \sum_{j=1}^{n} \alpha_j \left(X_j^2 + Y_j^2\right) = -\frac{1}{2} \langle \boldsymbol{\alpha}, \mathbf{R} \rangle \tag{17}$$

Theorem 2.2.[3] (Dirichlet) If the condition $\mathrm{A}_2^{\mathrm{n}}$ is satisfied and the numbers $\alpha_1, \dots, \alpha_{\mathrm{n}}$ are of the same sign, then the stationary point $\boldsymbol{\zeta} = 0$ is stable according to Lyapunov.

Here the role of the integral f is played by the Hamiltonian γ itself, for it is an integral, the notation (8) has the form (7) with $k = 2$ and the form $\gamma_2 = g_2 = -\frac{1}{2}\sum_{j=1}^{n} \alpha_j R_j = -\frac{1}{2}\langle \boldsymbol{\alpha}, \mathbf{R} \rangle$ is sign-defined, for $\mathbf{R} \geq 0$.

2.3. Formal stability

By formal, we will mean power series, about the convergence of which nothing is known.

Definition 2.3.[4] A stationary point (2) of a real Hamiltonian system (1) is *formally stable* if there exists a formal real sign-defined integral (10) of the system (1), i.e., the formal identity (11) is satisfied and the homogeneous form f_l is null only at $\boldsymbol{\zeta} = 0$.

Formal stability means that the departure of solutions from the stationary point, if anything, is very slow: slower than any finite degree of t.

Definition 2.4.[5] A stationary point (2) of a real Hamiltonian system (1) is *formally stable* if there exists a formal real integral

$$f(\boldsymbol{\zeta}) = f_l(\boldsymbol{\zeta}) + f_{l+1}(\boldsymbol{\zeta}) + \cdots + f_m(\boldsymbol{\zeta}) + \tilde{f}^{(m)}(\boldsymbol{\zeta})$$

of system (1), where $f_k(\boldsymbol{\zeta})$ are homogeneous forms of degree k and the sum

$$f^*(\boldsymbol{\zeta}) = f_l + f_{l+1} + \cdots + f_m \tag{18}$$

does not equal to zero in some neighborhood of the point $\boldsymbol{\zeta} = 0$ besides it.

Definition 2.5.[6] A point $\boldsymbol{\zeta}^0$ is called a *root of order* $\boldsymbol{k}$ of a polynomial $\hat{f}(\boldsymbol{\zeta})$, if at this point, the $\hat{f}$ itself and all its partial derivatives up to and including order k are zero, but at least one derivative of order $k+1$ is nonzero.

Conjecture 2.1. If a real polynomial (18) with $m > l$ does not converge to zero in some neighborhood of point $\boldsymbol{\zeta} = 0$ except it, then every real root $\boldsymbol{\zeta}^0$ of the polynomial f_l other than $\boldsymbol{\zeta} = 0$ has an even order.

Example 2.1. Let $\hat{f} = \xi^2 + \eta^4$. Then $f_2 = \xi^2$, $f_4 = \eta^4$. The equation $\hat{f} = 0$ has no real solutions except $\xi = \eta = 0$. The equation $f_2 \quad = 0$ has solutions $\xi = 0$, η is arbitrary and all these roots $\xi = 0, \eta \neq 0$ have order 2.

Since $r_j r_k = -\frac{1}{4} R_j R_k$, then under the condition A_4^n, the sum (9) takes the form

$$g = -\frac{1}{2}\langle \boldsymbol{\alpha}, \mathbf{R} \rangle - \frac{1}{4}\langle C\mathbf{R}, \mathbf{R} \rangle + \tilde{g}^{(5)} \tag{19}$$

Hence, all elements of matrix C are real.

Let $K \subset \mathbb{R}^n$ be a linear shell of integers $\mathbf{q}$ satisfying the equation $\langle \boldsymbol{\alpha}, \mathbf{q} \rangle = 0$, and $Q = \{\mathbf{q} \geq 0, \mathbf{q} \neq$

$0\} \subset \mathbb{R}^n$ is a non-negative orthant without origin.

Theorem 2.3.[7] If Condition A_4^n is satisfied and in (19)

$$\langle C\mathbf{q}, \mathbf{q}\rangle \neq 0 \text{ for } \mathbf{q} \in K \cap Q \tag{20}$$

then the point $\boldsymbol{\zeta} = 0$ is formally stable in the sense of Definition 2.3.

Here, the normal form of the Hamiltonian (5) from Theorem 2.1 is used to construct the formal integral.

According to (16) in real coordinates, the normal form (7) is

$$g = -\frac{1}{2}\langle \boldsymbol{\alpha}, \mathbf{R}\rangle + \sum_{l=2}^{[k/2]} h_l(\mathbf{R}) + \tilde{g}^{(k)} \tag{21}$$

where the homogeneous polynomials $h_l = (i/2)^l g_l(\mathbf{R})$ are real. The following generalization of Theorem 2.3 is proved verbatim like it.

Theorem 2.4. If the condition A_k^n is satisfied and in the normal form (21)

$$\sum_{l=2}^{[k/2]} h_l(\mathbf{R}) \neq 0 \text{ for } \mathbf{R} \in K \cap Q,$$

then the point $\boldsymbol{\zeta} = 0$ is formally stable in the sense of Definition 2.4.

This theorem is used implicitly by Markeev[5].

Markeev's condition 2[5]: System of equations

$$\langle \boldsymbol{\alpha}, \mathbf{q}\rangle = 0, \quad \langle C\mathbf{q}, \mathbf{q}\rangle = 0$$

has no solution $\mathbf{q} \in Q$, i.e., $\mathbf{q} \geq 0$, $\mathbf{q} \neq 0$.

Under conditions A_4^n and Markeev 2, the conditions of Theorem 2.3 are fulfilled and there is formal stability. But Markeev's condition 2 is easier to check than the (20) condition.

If $n = 2$, the Markeev's condition 2 takes the form: system of two equations

$$\alpha_1 q_1 + \alpha_2 q_2 = 0, \qquad c_{20} q_1^2 + 2c_{11} q_1 q_2 + c_{02} q_2^2 = 0$$

has no solution $q_1, q_2 \geq 0$ with $q_1 + q_2 \neq 0$.

But the solutions of the first equation have the form $q_1 = -\frac{\alpha_2}{\alpha_1} q_2$. For them, $q_1, q_2 > 0$ only when $\alpha_1\alpha_2 < 0$, i.e., the first equation has no solutions with $q_1, q_2 > 0$ and under Dirichlet Theorem 2.2 $\alpha_1\alpha_2 > 0$. Substituting them into the second equation and reducing by q_2^2/α_1^2, we obtain the condition

$$M_2 \stackrel{\text{def}}{=} c_{20}\alpha_2^2 - 2c_{11}\alpha_1\alpha_2 + c_{02}\alpha_1^2 \neq 0, \qquad \alpha_1\alpha_2 < 0 \tag{22}$$

which is called the Arnold-Moser condition.

Under this condition, there is not only formal stability, but also Lyapunov stability, because there are one-parameter families of two-dimensional invariant tori with similar sets of frequencies that lock the origin of coordinates. However, Moser[8] and Arnold[9] made mistakes in proving this fact. At the end of the article[8] is a criticism of the first proof by Moser[8]. This criticism consisted of the following. Moser proves that on every invariant surface $\gamma = c = \text{const}$, there is some stability zone $\| \boldsymbol{\zeta} \| < \varepsilon_0$. Thus, generally speaking, ε_0 can depend on c, i.e., $\varepsilon_0 = \varepsilon_0(c)$. It follows from his Theorem 9 that

$$\varepsilon_0(c) > 0 \tag{23}$$

for all sufficiently small c. Then he assumes that $\varepsilon_0(c)$ has a positive lower bound:

$$\varepsilon_0(c) > \varepsilon > 0 \tag{24}$$

But nowhere he proves it. From the property (24), it is indeed easy to deduce the stability of zero, which Moser does. At the same time, the proved property (23) is not sufficient for stability of zero if

$$\varliminf_{c\to 0} \varepsilon_0(c) = 0 \tag{25}$$

Siegel and Moser accounted for this criticism and gave the second correct proof in *Lectures on Celestial Mechanics*[10]. The criticism of the single proof by Arnold[9] is given in "Stability in a Hamiltonian system"[11]. But Arnold didn't take it into account and didn't correct his proof. He did, however, correct its formulation in "A letter to the editors"[12].

On page 86 of Markeev's book[5] is formulated:

Theorem 2.5. Let $n = 2$, the condition A_k^2 be satisfied, and in normal form (21) $\sum_{l=2}^{[k/2]} h_l(\alpha_2, -\alpha_1) \neq 0$, then the equilibrium position is stable according to Lyapunov.

The proof is given in appendix 2 of the report[13]. It repeats the reasoning by Moser in his *Lectures on Hamiltonian Systems*[8], which contains the error indicated above (see *Analytic form of differential equations (II)*[1]). Therefore, this theorem cannot be considered proven.

2.4. Theoretical-numerical analysis of frequencies

Many works on stability use conditions like Markeev's condition 2, where the number-theoretic character of frequencies α_j is not taken into account. And yet the structure of the normal form depends on them. For example, if the equation $\langle \boldsymbol{\alpha}, \mathbf{q} \rangle = 0$ has no solutions in integer $\mathbf{q} \neq 0$, then Condition A_∞^n is satisfied and the normal form of the Hamiltonian (5), (7) is $g(\mathbf{r})$. Then any r_j is a formal integral and the stationary point is formally stable. In particular, at $n = 2$, this is satisfied if the ratio α_1/α_2 is an irrational number.

Example 2.2. According to Markeev[5], the stability of the libration points of the planar circular restricted three-body problem is studied. There $\mathrm{n} = 2$, the frequencies $\omega_1 = \alpha_1$, $\omega_2 = -\alpha_2$ with $1 \geq \omega_1 > \omega_2 > 0$ satisfy the equation

$$\omega^4 - \omega^2 + \frac{27}{4}\mu(1-\mu) = 0 \tag{26}$$

where μ is the ratio of the masses of the two bodies and the only parameter of the problem ($0 \leq \mu \leq 1$). In this case, the stability is studied for

$$0 < \mu < 0.4 \tag{27}$$

It is shown in § 4, Ch. 7 of the book[5] that according to (4.7) in the normal form (2.17) $h_2(\alpha_2, -\alpha_1) = 0$ at

$$644\omega_1^4\omega_2^4 - 541\omega_1^2\omega_2^2 + 36 = 0 \tag{28}$$

Let us show that at these values the frequencies of ω_1 and ω_2 are incommensurable, i.e., formal stability takes place

Let's assume $\omega_1^2 = x$, $\omega_2^2 = y$, and note that by Vieta's formulae from (26) and (28) the equations follow

$$644x^2y^2 - 541xy + 36 = 0 \tag{29}$$

$$x + y = 1 \tag{30}$$

$$xy = \frac{27}{4}\mu(1-\mu) \tag{31}$$

From Equation (29) we get

$$xy = \frac{541 \pm \sqrt{199945}}{1288} \tag{32}$$

The product xy can have two values

$$(xy)_1 = 0.7671988\ldots, \qquad (xy)_2 = 0.0728632\ldots$$

But on the interval (27), the function $\mu(1-\mu)$ takes the largest value at the right end at $\mu = 0.4$. There $27\mu(1-\mu)/4 = 0.2592\ldots$. Therefore, it follows from equality (31) that

$$xy = (xy)_2 = \frac{541 - \sqrt{199945}}{1288} \stackrel{\text{def}}{=} \Omega \tag{33}$$

Assume $z = x/y$, i.e., $x = zy$. Here z is the ratio of the squares of the frequencies. According to (30), we get $y = 1/(z+1)$. Substituting this and $x = zy$ in (33), we get $z/(1+z^2) = \Omega$. Consequently, z satisfies the quadratic equation $(z+1)^2 = z/\Omega$. Its roots are $z = \left(1 - 2\Omega \pm \sqrt{1-4\Omega}\right)/(2\Omega)$. Given (33), we see that both values of z are irrational. Consequently, the ratio of frequencies $\sqrt{z}$ is also irrational.

In Markeev[5], the following are used to prove stability in this case: the unproved Theorem 2.5 and the cumbersome calculation of the coefficients of the sixth-order terms of the normal form of the Hamiltonian.

Example 2.3. In Markeev[5], the stability of libration points of a spatial circular restricted three-body problem is studied. There $\mathrm{n} = 3$, the frequencies ω_1 and ω_2 are the same as in Example 2.2, and $\omega_3 = 1$. In Chapter 8, § 3 on page 136 of the book[5], there the formal stability theorem is formulated for all values of μ such that $0 < 27\mu(1-\mu) < 1$, except where there is double resonance. Let us show that in this problem the double resonance is impossible.

Indeed, in the case of double resonance, the frequencies ω_1 and ω_2 are commensurate with each other and commensurate with unity. Let $\omega_1 = r/s$, $\omega_2 = p\omega_1/q$, where p, q, r, s are integers,

$$0 < p < q, \qquad 0 < r < s \tag{34}$$

According to (30), $\omega_1^2 + \omega_2^2 = 1$, that is, $\frac{r^2}{s^2}\left(1 + \frac{p^2}{q^2}\right) = 1$, or $1 + \frac{p^2}{q^2} = \frac{s^2}{r^2}$, or

$$q^2r^2 + p^2r^2 = s^2q^2 \tag{35}$$

Let's put

$$k = qr, \qquad l = pr, \qquad m = qs \tag{36}$$

Then the Equation (35) takes the form

$$k^2 + l^2 = m^2 \tag{37}$$

As we know, all solutions to the Equation (37) in integer non-negative numbers have the form

$$k = \kappa^2 - 1, \qquad l = 2\kappa, \qquad m = \kappa^2 + 1 \tag{38}$$

where κ is a non-negative integer. According to (34) and (36), $l < k$. Therefore, the Equation (38) will apply when $\kappa > 2$, and when $\kappa = 0$, $\kappa = 1$ and $\kappa = 2$, we put

$$k = 2\kappa, \qquad l = \kappa^2 - 1, \qquad m = \kappa^2 + 1. \tag{39}$$

By direct verification, we make sure that when $0 \le \kappa < 3$, the Equations (36) and (39) are impossible for integers. When $\kappa > 3$, the equations

$$q = \frac{\kappa^2 + 1}{s}, r = \frac{2\kappa}{p}, qr = \kappa^2 - 1 = \frac{(\kappa^2 + 1)2\kappa}{ps}$$

are following from the Equations (36) and (38). Therefore,

$$ps = \frac{2\kappa(\kappa^2 + 1)}{(\kappa + 1)(\kappa - 1)}. \tag{40}$$

The numbers $\kappa - 1$, κ, $\kappa + 1$ have no common factor, and the numbers $\kappa^2 + 1$ and $\kappa + 1$ have no common factor other than 2. Therefore, the ratio (40) cannot be an integer.

2.5. Formal stability investigation in a generic case of three degrees of freedom

Earlier the second author[14] proposed a schematic description of a method for studying formal stability of the stationary point of a Hamiltonian system. This method is based on the following key results:

- normal form of the Hamiltonian system in the neighborhood of the stationary point;
- formal stability Theorem 2.3;
- q-analogs of classical objects of elimination theory[15].

The drawback of this approach is that it does not take into account multi-frequency resonances of order three or more, which appear in systems with more than two degrees of freedom.

Below we describe a method for investigating formal stability of the equilibrium position for a multiparameter Hamiltonian system with three degrees of freedom. Consider a Hamiltonian system in the vicinity of the equilibrium position for which the following conditions are satisfied:

- the number of degrees of freedom of the system is greater than two,
- the quadratic form γ_2 in expansion (3) is nondegenerate and is not sign definite,
- the Hamiltonian function γ smoothly depends of the vector of parameters $\mathbf{P}$ from a domain $\Pi \subset \mathbb{R}^m$.

Corollary 2.1 (of Theorem 2.3). If in $\mathbb{R}^3$, the intersection of the plane $\langle \boldsymbol{\lambda}, \mathbf{q} \rangle = 0$ and the cone $\langle C\mathbf{q}, \mathbf{q} \rangle = 0$ either does not lie in $\mathbf{Q} = \mathbb{R}^3_+$, or lies in $\mathbf{Q} = \mathbb{R}^3_+$, but does not contain the integer vector $\mathbf{q}$, then the stationary point is formally stable.

The behavior of the phase flow in the first approximations is described by the linear Hamiltonian system

$$\dot{\boldsymbol{\zeta}} = B(\mathbf{P})\boldsymbol{\zeta}, \qquad B(\mathbf{P}) = \frac{1}{2} J \frac{\partial^2 \gamma_2(\mathbf{P})}{\partial \boldsymbol{\zeta}^2} \tag{41}$$

where J is the symplectic unit matrix. The characteristic polynomial $\breve{f}(\boldsymbol{\lambda})$ of the matrix $B(\mathbf{P})$ contains only even degrees of λ; therefore, it is a polynomial of $\mu = \lambda^2$. According to Batkhin et al.[16], such a polynomial is called semi-characteristic:

$$f_n(\mu) = \sum_{k=0}^{n} f_{n-k}(\mathbf{P})\mu^k, \qquad f_0 \equiv 1 \tag{42}$$

Definition 2.6. The *stability set* Σ of the linear system (41) is the set of all values of parameters $\mathbf{P} \in \Pi$ for which the stationary point $\boldsymbol{\zeta} = 0$ is Lyapunov stable.

In order to apply Theorem 2.3 on formal stability, we should find the boundaries of the domains in the space of parameters Π determined by the resonant varieties corresponding to strong resonances.

Definition 2.7. A resonant variety $\mathcal{R}_n^{\mathbf{p}}$ in the space K of coefficients $a_1, \dots, a_n$ of the semi-characteristic polynomial $f_n(\mu)$ of degree n is an algebraic variety on which the vector of basis eigenvalues $\boldsymbol{\lambda}$ of the corresponding characteristic polynomial $\check{f}(\lambda)$ is a nontrivial solution of the resonant Equation (6) for a fixed integer vector $\mathbf{p} = \mathbf{p}^*$. An analytical representation of the variety $\mathcal{R}_n^{\mathbf{p}*}$ in an implicit or parametric form is denoted by $R_n^{\mathbf{p}*}$.

To examine the formal stability of a stationary point of a Hamiltonian system (1), we should find in the space of parameters Π the stability set Σ of the linear system (41), find such domains, in which the quadratic form $\gamma_2(\mathbf{z})$ is not sign definite, find parts S_k in these domains that do not contain strong resonances, normalize the Hamiltonian in each of these parts S_k up to order four, and then apply Theorem 2.3. To do this, it is sufficient to select a point in each S_k in the space of parameters and use one of the normalization algorithms for the Hamiltonian function. Since all eigenvalues λ_k $(k = 1, \dots, n)$ are simple at each interior point of S_k, the invariant normalization algorithm can be easily applied. For $n = 3$, the borders between the parts S_k are defined by the following resonant varieties: $\mathcal{R}_3^{(2,1,0)}$, $\mathcal{R}_3^{(3,1,0)}$ corresponding two-frequency resonances and $\mathcal{R}_3^{(1,1,1)}$, $\mathcal{R}_3^{(2,1,1)}$ corresponding three-frequency resonances.

A general description of the procedure for obtaining condition on the existence of two and multi-frequency resonances is as follows (for details see "Calculation of a strong resonance condition in a Hamiltonian system"[17]):

(1) For a certain vector $\mathbf{p}^* = (r, q, 1)$, where $r, q \in \mathbb{Q}$, $r, q \neq 0$, satisfying the resonance Equation (6), a polynomial ideal is composed $\mathcal{J} = \{\langle \mathbf{p}^*, \boldsymbol{\lambda} \rangle, \lambda_j^2 - \mu_j\}$;

(2) Gröbner basis $\mathcal{G}$ of this ideal with the elimination monomial order of variables $\lambda_j, \mu_j, j = 1, \dots, n$ is computed. The first polynomial $R_3^{(r,q,1)}(\mu_j)$ of $\mathcal{G}$ is a quasi-homogeneous polynomial in the variables $\mu_j, j = 1, \dots, n$. Its zeroes determine the condition of existence of resonance for a given vector $\mathbf{p}^*$.

This condition takes the form

$$\begin{aligned} R_3^{(r,q,1)}(\mu_j) \equiv q^4\mu_2^2 - 2q^2r^2\mu_1\mu_2 + r^4\mu_1^2 - \\ -2q^2\mu_2\mu_3 - 2r^2\mu_1\mu_3 + \mu_3^2 = 0. \end{aligned} \tag{43}$$

For condition (43) a power transformation[18], defined by a matrix $M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ with the corresponding variable change

$$\mu_1 = s_2 s_3, \qquad \mu_2 = s_1 s_3, \qquad \mu_3 = s_3$$

is done. It reduces the polynomial $R_3^{(r,q,1)}(\mu_j)$ into a polynomial of two variables

$$\begin{aligned} \tilde{R}_3^{(r,q,1)} \equiv q^4 s_1^2 - 2q^2r^2 s_1 s_2 + r^4 s_2^2 - \\ -2q^2 s_1 - 2r^2 s_2 + 1 = 0, \end{aligned} \tag{44}$$

which has the parametric representation of the roots

$$\begin{aligned} \mu_1 = &\quad (r^2u(q+1)+q-1)^2v, \\ \mu_2 = &\quad (r^2u-1)^2vr^2, \\ \mu_3 = &\quad (r^2u+2q-1)^2vr^2. \end{aligned} \tag{45}$$

For each strong resonance of orders 2, 3 and 4 parametric representation of the corresponding variety was obtained. Their mutual location is shown in **Figure 1**.

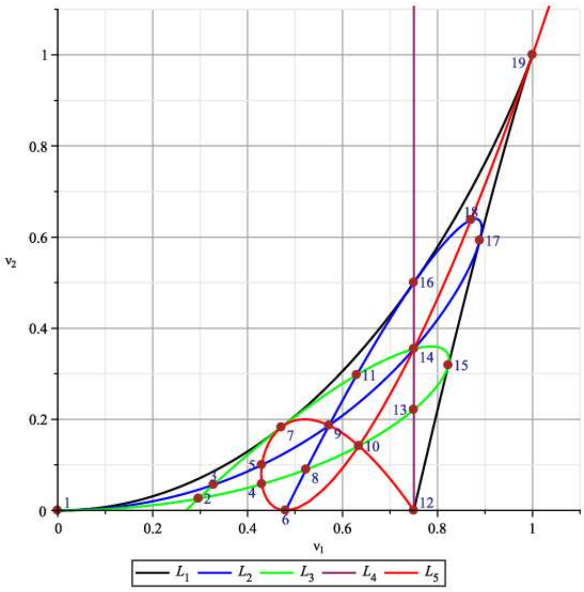

Figure 1. Resonant varieties in parametric variables.

Curve L_1 (black) is variety $\mathcal{R}_3^{(1,1,0)}$, curve L_2 (blue) is variety $\mathcal{R}_3^{(2,1,0)}$, curve L_3 (green) is variety $\mathcal{R}_3^{(3,1,0)}$, line L_4 (magenta) is variety $\mathcal{R}_3^{(1,1,1)}$ and curve L_5 (red) is variety $\mathcal{R}_3^{(2,1,1)}$. Each point at **Figure 1** denotes the set of parameters for which the multiplicity of resonance changes from 1 to 2.

The curve L_1 plays a special role, it determines the boundary of the domain of stability Σ of the stationary point in linear approximation. This curve is the image of the discriminant set $\mathcal{D}(f_3)$, which divides the space of coefficients of the cubic polynomial into two parts. In one part, all roots of the polynomial are real, and in the other part there is a pair of complex conjugate roots and one real root. The curvilinear triangle is the boundary of the domain Σ. The other resonant curves are completely or partially lay within this domain. Note that **Figure 1** is slightly similar to the figure 14 in *Libration Points in Celestial Mechanics and Cosmo Dynamics*[5].

Example 2.4. Consider a modified Hamiltonian oscillation system with three degrees of freedom and two parameters. Such a system arises in the study of motion near a stable equilibrium position of three mathematical pendulums of equal length l and close masses, connected by weightless elastic springs of stiffness k. If the normal coordinates $Q=(Q_1,Q_2,Q_3)$ are chosen as the generalized coordinates, then the quadratic part of the Hamilton function is written as

$$\begin{aligned} H_2 = &-\frac{(2\alpha+1)Q_1^2}{2\alpha} - \frac{\alpha P_1^2}{4\alpha+2} + (\beta+1)Q_2^2 + \frac{P_2^2}{4} \\ &-(2\alpha+1)(2\beta\alpha+\beta+1)Q_3^2 - \frac{P_3^2}{8\alpha+4}, \end{aligned} \tag{46}$$

where α and β are the parameters which, according to the physical meaning of the problem, must be positive. Since the form H_2 is not sign-defined, the Dirichlet theorem is inapplicable. Let us perform a study of the formal stability of the equilibrium position.

For the initial parameters, the vector of basic eigenvalues is the following $\boldsymbol{\lambda} = \left(-1, \sqrt{\beta+1}, -\sqrt{2\beta\alpha+\beta+1}\right)^T$. Let us introduce new parameters a, b so that the value of a is the square of the deviation of the second frequency from 1, and the value of b is the square of the deviation of the third frequency from the second frequency, i.e.,

$$\alpha = \frac{b}{2a}, \qquad \beta = a. \tag{47}$$

In the new parameters a, b, the domain $\mathcal{K} \subset \Pi$, for values of which all eigenvalues are purely imaginary, is a positive quadrant of the parameter plane Π, and the vector

$$\boldsymbol{\lambda} = (-1, \lambda_2, -\lambda_3)^T, \tag{48}$$

where $\lambda_2 = \sqrt{a+1}$, $\lambda_3 = \sqrt{a+b+1}$.

The expansion of the Hamiltonian up to the 4th order in the neighborhood of the equilibrium position gives the following forms:

$$\begin{aligned} H_2 = \ & -\frac{(a+b)Q_1^2}{b} - \frac{bP_1^2}{4(a+b)} + \lambda_2 Q_2^2 + \frac{P_2^2}{4} \\ & -\frac{(a+b)\lambda_3 Q_3^2}{a} - \frac{aP_3^2}{4(a+b)}, \end{aligned} \tag{49}$$

$$\begin{aligned} H_4 = \ & -\frac{(a+b)Q_1^4}{12b} + \frac{(1+2a)Q_1^2Q_2^2}{2} \\ & -\frac{(a+b)(2b+2a+1)Q_1^2Q_3^2}{2a} \\ & +(b+3a+1)Q_1Q_2^2Q_3 - \frac{(4a+1)Q_2^4}{12} \\ & -\frac{(a^2-b^2)(3b+3a+1)Q_1Q_3^3}{3a^2} \\ & +\frac{(2b+1+4a)Q_2^2Q_3^2}{2} \\ & -\frac{(a+b)(a^2-ab+b^2)(4b+4a+1)Q_3^4}{12a^3}. \end{aligned} \tag{50}$$

For application of Theorem 2.3 on the formal stability, it is necessary to normalize the Hamiltonian to the 4th order in the domain without strong resonances.

Let us determine the location of resonance manifolds in the domain of $\mathcal{K}$. Since there is no third degree form in the original Hamiltonian, it is sufficient to study the 4th order resonances, i.e., the manifolds $\mathcal{R}_3^{\mathbf{p}_1^*}$, $\mathbf{p}_1^* = (3,1,0)$ and $\mathcal{R}_3^{\mathbf{p}_2^*}$, $\mathbf{p}_2^* = (2,1,1)$. In variables a, b, these manifolds are written in the following form:

$$\begin{aligned} R_3^{\mathbf{p}_1^*} = \ & (9a+8)(8-a)(9a+8+9b) \\ & \times (8a+8-b)(8a+8+9b)(a-8+b) = 0, \\ R_3^{\mathbf{p}_2^*} = \ & (9a^2-6ab+b^2+8a-8b)(b^2-16a-8b) \\ & \times (9a^2+24ab+16b^2+8a+16b) = 0. \end{aligned}$$

Obviously, the manifold corresponding to the two-frequency resonances on the first quadrant $a, b > 0$ is the union of three lines:

$$a = 8, \qquad b = 8(a+1), \qquad a+b = 8.$$

The manifold corresponding to the three-frequency resonances on the first quadrant $a, b > 0$ is the union of two parabolas

$$(3a-b)^2 + 8(a-b) = 0,$$
$$(b-4)^2 - 16(a+1) = 0.$$

On the described manifolds the structure of the normal form changes, and the stability study can be carried out according to the methods in *Libration Points in Celestial Mechanics and Cosmo Dynamics*[5]. Here we provide formal stability investigation of the initial system in accordance with the conditions of Theorem 2.3 and Corollary 2.1, i.e., in the domain of parameter values where there are no strong resonances.

At the first step we normalize the quadratic part of H_2, and in the new variables $\boldsymbol{\xi}, \boldsymbol{\eta}$ we obtain the Hamiltonian $h = h_2 + h_4$ in the form of

$$h_2 = -\frac{1}{2}(\xi_1^2 + \eta_1^2) + \frac{\lambda_2}{2}(\xi_2^2 + \eta_2^2) - \frac{\lambda_3}{2}(\xi_3^2 + \eta_3^2), \tag{51}$$

$$\begin{aligned} h_4 = & -\frac{b\xi_1^4}{48a+48b} - \frac{(4a+1)\xi_2^4}{48a+48} \\ & -\frac{(a^2-ab+b^2)(b+a+1/4)\xi_3^4}{12(a+b)a(b+a+1)} \\ & +\frac{(2b+1+4a)a\xi_2^2\xi_3^2}{8\lambda_2\lambda_3(a+b)} \\ & -\frac{(a-b)(3b+3a+1)\sqrt{b}\xi_1\xi_3^3}{12(a+b)\lambda_3^{3/4}\sqrt{a}} \\ & +\frac{(b+3a+1)\sqrt{ba}\xi_1\xi_2^2\xi_3}{4\lambda_2\lambda_3^{1/4}(a+b)} \\ & -\frac{(2b+2a+1)b\xi_1^2\xi_3^2}{8\lambda_3(a+b)} + \frac{(1+2a)b\xi_1^2\xi_2^2}{8\lambda_2(a+b)}. \end{aligned} \tag{52}$$

At the second step, we perform the normalization using the Zhuravlev invariant normalization method (for details see Zhuravlev et al.[19]). For this purpose, we carry out the complexification of the real Hamiltonian of valence $2i$ with the help of substitution

$$\xi_j = \frac{1}{2i}(Z_j - \bar{Z}_j), \qquad \eta_j = \frac{1}{2}(Z_j + \bar{Z}_j), \qquad j = 1,2,3.$$

The quadratic form of h_2 will take the form

$$\tilde{h}_2 = -iZ_1\bar{Z}_1 + i\lambda_2 Z_2\bar{Z}_2 - i\lambda_3 Z_3\bar{Z}_3,$$

and defines the unperturbed solutions in the form

$$Z_j = X_j \exp(i\lambda_j t), \qquad \bar{Z}_j = \bar{X}_j \exp(-i\lambda_j t), \qquad j = 1,2,3.$$

Averaging $\tilde{h}_2$ along the unperturbed solution, we obtain the following term of the normalized form. Due to the fact that the normalization is performed under the condition that there are no strong resonances, the obtained NF depends only on the products $X_j\bar{X}_j$, $j = 1,2,3$, which are the action variables $\rho_j = iX_j\bar{X}_j$, and it will be written as $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4$, where

$$\begin{aligned}
\mathcal{H}_2 = \;& -\rho_1 + \lambda_2\rho_2 - \lambda_3\rho_3, \\
\mathcal{H}_4 = \;& -\frac{b\rho_1^2}{128(a+b)} + \frac{b(1+2a)\rho_1\rho_2}{32\lambda_2(a+b)} \\
& -\frac{b(2b+2a+1)\rho_1\rho_3}{32\lambda_3(a+b)} - \frac{(4a+1)\rho_2^2}{128\lambda_2} \\
& +\frac{a(2b+1+4a)\rho_2\rho_3}{32\lambda_2\lambda_3(a+b)} \\
& -\frac{(a^2-ab+b^2)(4b+4a+1)\rho_3^2}{128(a+b)a\lambda_3^2}.
\end{aligned} \tag{53}$$

According to Corollary 2.1 of Theorem 2.3, we find in the first quadrant of space Π of the new parameters a, b of the domains in which the conditions of Corollary are satisfied. From the equation of the plane $\mathcal{H}_2 = 0$, we express the variable ρ_1, substitute it into the equation of the cone $\mathcal{H}_4 = 0$ and obtain the quadratic equation

$$g \equiv a_0\zeta^2 + a_1\zeta + a_2 = 0, \quad \zeta = \rho_2/\rho_3, \tag{54}$$

whose coefficients are the following

$$\begin{aligned}
a_0 = \;& \frac{G_0}{128(a+b)\lambda_2}, \\
a_1 = \;& -\frac{G_1}{64\lambda_2\lambda_3(a+b)}, \\
a_2 = \;& \frac{G_2}{128(a+b)a\lambda_3^2}, \\
G_0 = \;& 7a^2b - 4a^2 + 6ab - a + 2b, \\
G_1 = \;& 7a^2b + 7ab^2 - 8a^2 + 6ab + 5b^2 - 2a + 3b, \\
G_2 = \;& 7a^3b + 14a^2b^2 + 7ab^3 - 4a^3 + 10a^2b + 10ab^2 \\
& -4b^3 - a^2 + 4ab - b^2
\end{aligned} \tag{55}$$

Conditions of Corollary 2.1 are satisfied in one of the following cases:

1) discriminant of Equation (54) is negative: $D(g) < 0$;
2) discriminant of Equation (54) is positive, but both roots are negative, so $D(g) > 0$, $a_1/a_0 > 0$, $a_2/a_0 \geq 0$;
3) there is at least one positive root of ζ^+, but the value of ρ_1 corresponding to it is non-positive, so $a_1/a_0 < 0$, $\zeta^+ < \lambda_3/\lambda_2$;
4) Equation (54) degenerates into a linear equation with a positive root ζ^+ and with $\rho_1 < 0$, so $a_0 = 0$, $\zeta^+ < \lambda_3/\lambda_2$;
5) given $D(g) = 0$, the multiple root of Equation (54) is $\zeta < 0$, or $\zeta > 0$ but $\rho_1 < 0$.

Note that cases 4 and 5 are realized only on curves, not in domains.

Discriminant of Equation (54) is the following:

$$\begin{aligned}
D(g) = \;& -\frac{4\lambda_2^2 G_3}{a\lambda_3^2 G_0^2}, \\
G_3 = \;& 56a^5b + 84a^4b^2 - 28a^3b^3 - 56a^2b^4 - 48a^5 \\
& +46a^4b + 78a^3b^2 - 12a^2b^3 - 35ab^4 - 24a^4 \\
& +32a^3b + 22a^2b^2 - 12ab^3 - 8b^4 - 3a^3 \\
& +6a^2b - 2b^3.
\end{aligned} \tag{56}$$

Case 1. It is obvious that in the first quadrant the sign of the discriminant can change only on the curve $G_3 = 0$. This curve divides the first quadrant of the plane (a, b) into three curvilinear segments marked I, II, III in **Figure 2**.

Substituting the coordinates of points from these domains shows that the discriminant is negative in domain II.

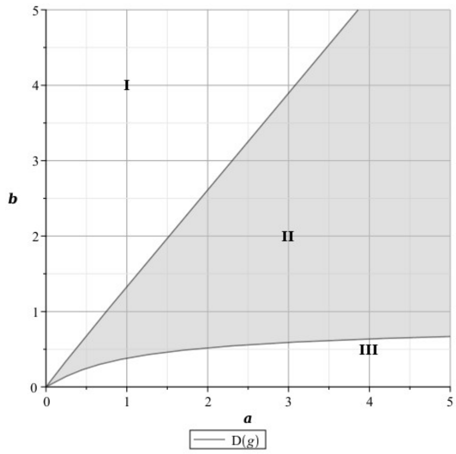

Figure 2. Domain of formal stability for case 1.

Case 2. Let us write out expressions for the ratios of the coefficients of the polynomial g

$$\frac{a_1}{a_0} = -\frac{2\lambda_2 G_1}{\lambda_3 G_0}, \qquad \frac{a_2}{a_0} = \frac{\lambda_2^2 G_2}{a\lambda_3^2 G_0}.$$

The signs of the ratios can change on the curves $G_j = 0,\ j = 0, 1, 2$. Their mutual arrangement is shown in **Figure 3** together with the curve $G_3 = 0$. In the mentioned domains where $D(g) > 0$, the values of $a_1/a_0, a_2/a_0$ are calculated. Both coefficients are positive in the domain VI.

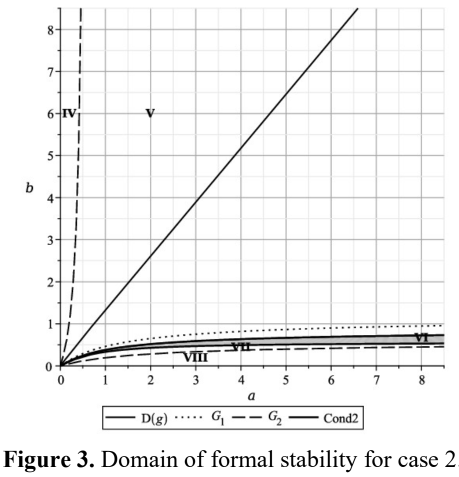

Figure 3. Domain of formal stability for case 2.

Case 3. The condition requires that there be at least one positive root $(a_1/a_0 < 0)$ and that the largest positive root $\zeta^+ = (-a_1/a_0 + \sqrt{D(g)})/2$ be less than the ratio λ_2/λ_3. This condition can be rewritten as $\left(2\frac{\lambda_2}{\lambda_3} + \frac{a_1}{a_0}\right)^2 - D(g) > 0$. In variables a, b this condition is rewritten as $\frac{4\lambda_2^2 G_4}{a\lambda_3^2 G_2} > 0,\ G_4 = 7ab^3 + 8a^3 + 4a^2b - 4b^3 + 2a^2 - b^2$. Two domains satisfy Condition 3 but one of them coincides with the domain VI from **Figure 3**. **Figure 4** shows the domain bounded by the curves $G_4 = 0$ and the discriminant curve $G_3 = 0$.

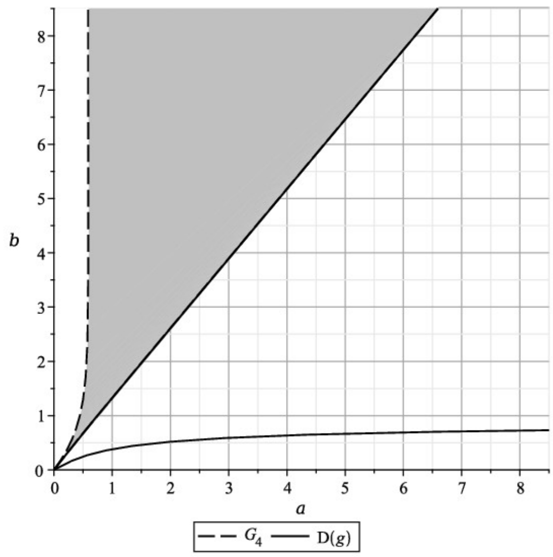

Figure 4. Domain of formal stability for case 3.

Cases 4 and **5** are always satisfied, since in these cases the only root of ρ_1 is always negative.

So, the final result is the domain shown in **Figure 5**. It is contained between the curves G_0 and G_4. The resonance varieties $\mathcal{R}_3^{p_1}$ (shown by dotted lines) and $\mathcal{R}_3^{p_1}$ (shown by solid lines) should be removed from this region.

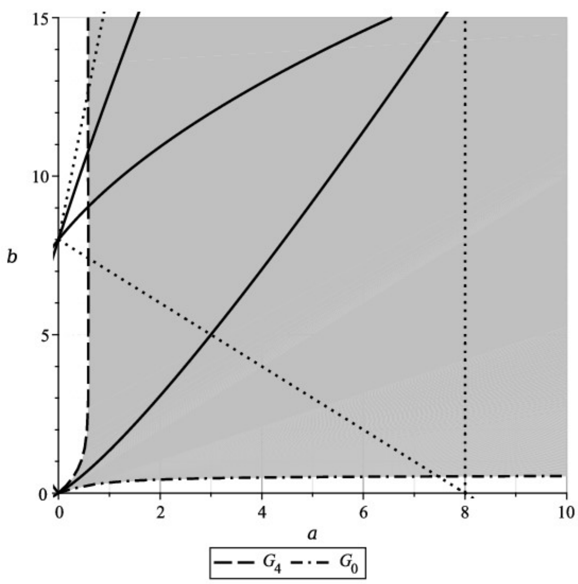

Figure 5. Final domain of formal stability.

2.6. Scattering order of solution

Let the function $f(t)$ be defined at real $t \to -\infty$. It is said to have order $\delta = \delta(t)$ if $\delta = \inf \varepsilon$ such that $f(t)/(-t)^{\varepsilon} \to 0$ at $t \to -\infty$. If $\delta > 0$, then $f(t)$ is unbounded, if $\delta < 0$, then $f(t) \to 0$ at $t \to -\infty$. In the latter case $\delta(f) < 0$, the larger δ is, the slower $f(t)$ approaches zero.

Definition 2.8. Let the solution $\boldsymbol{\zeta}(t)$ of the Hamiltonian system (1) tend to a stationary point (2) at $t \to -\infty$. On this solution *order of scattering* $\Delta = \min\{\delta \parallel \boldsymbol{\zeta} \parallel\}$.

Definition 2.9. The *scattering order of solutions* of the system (1) from the stationary point (2) $\tilde{\Delta}$ is the lower bound of the scatter order Δ over all solutions $\boldsymbol{\zeta}(t)$ that tend to the point (2) at $t \to -\infty$.

The smaller $\tilde{\Delta} < 0$, the faster the solutions are scattered from the stationary point. At formal stability the order of scattering of solutions from the stationary point is zero. Let us estimate the order of scattering $\tilde{\Delta}$ in the absence of formal stability. The cases $-10^{-10} < \tilde{\Delta} < 0$ can be considered as weak stable.

Conjecture 2.2. Let the condition A_2^n and $\kappa = min \parallel \mathbf{p} + \mathbf{q} \parallel > 2$ by integer solutions $\mathbf{p} \geq 0$, $\mathbf{q} \geq 0$ of equation $\langle \boldsymbol{\alpha}, \mathbf{p} - \mathbf{q} \rangle = 0$ be satisfied, then the order of scatter of the system solutions (1) from the stationary point $\tilde{\Delta} \geq (2 - \kappa)^{-1}$.

Example 2.5. Consider a real Hamiltonian with n degrees of freedom in complex coordinates

$$G = g_{\mathbf{pq}}\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}} + g_{\mathbf{qp}}\mathbf{x}^{\mathbf{q}}\mathbf{y}^{\mathbf{p}} \tag{57}$$

where integer $\mathbf{p}, \mathbf{q} > 0$, $\parallel \mathbf{p} + \mathbf{q} \parallel \overset{\text{def}}{=} \kappa > 2$, all differences

$$p_j - q_j \neq 0 \text{ have one sign } \sigma = sign(p_j - q_j), \qquad j = 1, \ldots, n, \tag{58}$$

all $p_j, q_j \neq 0$,

$$\langle \boldsymbol{\alpha}, \mathbf{p} - \mathbf{q} \rangle = 0, \tag{59}$$

and the complex coefficients $g_{\mathbf{pq}}$ and $g_{\mathbf{qp}}$ are related by the relations (20), i.e.,

$$g_{\mathbf{pq}} = (-i)^{\kappa}\bar{g}_{\mathbf{qp}} \tag{60}$$

and will be defined later.

The Hamiltonian system corresponding to the Hamiltonian (57) is

$$\begin{aligned} \dot{x}_j &= q_j g_{\mathbf{pq}}\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}-\mathbf{e}_j} + p_j g_{\mathbf{qp}}\mathbf{x}^{\mathbf{q}}\mathbf{y}^{\mathbf{p}-\mathbf{e}_j}, \\ \dot{y}_j &= -p_j g_{\mathbf{pq}}\mathbf{x}^{\mathbf{p}-\mathbf{e}_j}\mathbf{y}^{\mathbf{q}} - q_j g_{\mathbf{qp}}\mathbf{x}^{\mathbf{q}-\mathbf{e}_j}\mathbf{y}^{\mathbf{p}}, \\ j &= 1, \ldots, n, \end{aligned} \tag{61}$$

where $\mathbf{e}_j$ is the jth orth. Multiplying the upper equation by y_j and the lower equation by x_j, we obtain the system

$$\begin{aligned} y_j\dot{x}_j &= q_j g_{\mathbf{pq}}\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}} + p_j g_{\mathbf{qp}}\mathbf{x}^{\mathbf{q}}\mathbf{y}^{\mathbf{p}}, \\ x_j\dot{y}_j &= -p_j g_{\mathbf{pq}}\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}} - q_j g_{\mathbf{qp}}\mathbf{x}^{\mathbf{q}}\mathbf{y}^{\mathbf{p}}, \end{aligned}$$

$$j = 1, \ldots, n. \tag{62}$$

Let's find a solution to this system of the form

$$x_j = A_j(-t)^{\Omega}, \qquad y_j = iA_j(-t)^{\Omega}, \qquad j = 1, \ldots, n, \tag{63}$$

where A_j are real positive constants, and Ω is a real exponent of degree and real $t < 0$. The solution of (63) has properties (13) characteristic of real solutions in complex coordinates. For the solution of (63) the Equation (62) take the form

$$\begin{aligned} -\Omega A_j^2\, i(-t)^{2\Omega-1} &= \left[q_j g_{\mathbf{pq}}(i)^{\|\mathbf{q}\|} + p_j g_{\mathbf{qp}}(i)^{\|\mathbf{p}\|}\right]\mathcal{A}, \\ -\Omega A_j^2\, i(-t)^{2\Omega-1} &= \left[p_j g_{\mathbf{pq}}(i)^{\|\mathbf{q}\|} + q_j g_{\mathbf{qp}}(i)^{\|\mathbf{p}\|}\right]\mathcal{A}, \\ \mathcal{A} &\stackrel{\mathrm{def}}{=} \mathbf{A}^{\mathbf{p}+\mathbf{q}}(-t)^{\Omega\kappa}, \qquad j = 1, \dots, n. \end{aligned} \tag{64}$$

Comparing the degrees of $-t$ in the Equation (64), we get the equality $2\Omega - 1 = \Omega\kappa$. It follows that $\Omega = (2-\kappa)^{-1}$, according to Conjecture 2.2. Now notice that in the pair of Equation (64) for one j the left-hand sides are equal. Therefore, by subtracting the lower equation from the upper equation and reducing by $\mathbf{A}^{\mathbf{p}+\mathbf{q}}(-t)^{\Omega\kappa}$, we obtain a system of equations

$$(p_j + q_j)\left[(i)^{\|\mathbf{q}\|} g_{\mathbf{pq}} + (i)^{\|\mathbf{p}\|} g_{\mathbf{qp}}\right] = 0, \qquad j = 1, \dots, n$$

which reduces to a single equation

$$\left[(i)^{\|\mathbf{q}\|} g_{\mathbf{pq}} + (i)^{\|\mathbf{p}\|} g_{\mathbf{qp}}\right] = 0. \tag{65}$$

According to (60) $g_{\mathbf{qp}} = (-i)^{\kappa}\bar{g}_{\mathbf{pq}}$. Therefore, this equation takes the form

$$(i)^{\|\mathbf{q}\|}\left[g_{\mathbf{pq}} + (-1)^{\|\mathbf{q}\|}\bar{g}_{\mathbf{pq}}\right] = 0. \tag{66}$$

Let's put

$$g_{\mathbf{pq}} = \begin{pmatrix} \tau = \pm 1, & \text{if } \| \mathbf{q} \| \text{ odd,} \\ \tau = i, & \text{if } \| \mathbf{q} \| \text{ even.} \end{pmatrix} \tag{67}$$

In both cases, the square bracket in (66) is cancelled. We will specify the value of $\tau = \pm 1$ later. Now the system (64) reduces to the system

$$\frac{iA_j^2}{\kappa - 2} = (i)^{\|\mathbf{q}\|}(q_j - p_j) g_{\mathbf{pq}} \mathbf{A}^{\mathbf{p}+\mathbf{q}}, \qquad j = 1, \dots, n. \tag{68}$$

According to (58), all differences $q_j - p_j$ have the sign $-\sigma$. Choose in (67) $\tau = \pm 1$ such that

$$(i)^{\|\mathbf{q}\|-1}(-\sigma) g_{\mathbf{pq}} = 1. \tag{69}$$

Indeed, if $\| \mathbf{q} \|$ is odd, then $(i)^{\|\mathbf{q}\|-1} = \pm 1$ and we can pick the sign τ so that there is equality (69). If $\| \mathbf{q} \|$ is even, then $(i)^{\|\mathbf{q}\|-1} g_{\mathbf{pq}} = \pm 1$ and we can pick the sign τ so that there is equality (69). Now the system (68) takes the form

$$\frac{A_j^2}{|p_j - q_j|} = (\kappa - 2)\mathbf{A}^{\mathbf{p}+\mathbf{q}}, \qquad j = 1, \dots, n. \tag{70}$$

Let us show that this system has a unique solution $\mathbf{A} > 0$. Let's go to logarithms. Then the system (70) will take the form

$$\ln \frac{A_j^2}{|p_j - q_j|} = \ln(\kappa - 2) + \sum_{k=1}^{n} (p_k + q_k)\ln A_k, j = 1, \dots, n. \tag{71}$$

This is a linear inhomogeneous system with respect to $\ln A_j$. Its determinant is $D = (-2)^{n-1}(\kappa - 2)$. This is easily proved by induction on n. Therefore, the system (71) has the unique solution $\ln\mathbf{A}$, i.e., the system (70) has the only solution $\mathbf{A} > 0$. According to (68) $A_j^2 = (p_j - q_j)\mathrm{const}$, $j = 1, \dots, n$. On these solutions $\sum_{j=1}^{n} \alpha_j x_j(t) y_j(t) = i \sum_{j=1}^{n} \alpha_j A_j^2 (-t)^{2\Omega} = i(-t)^{2\Omega}\langle \boldsymbol{\alpha}, \mathbf{p} - \mathbf{q}\rangle\mathrm{const}$. According to (59), this sum is identically equal to zero. Therefore, the obtained solution (63) is also the solution of the system with the Hamiltonian $g = i\sum_{j=1}^{n} \alpha_j x_j(t) y_j(t) + G$, which is a special case of the complex normal form.

Remark 2.1. The example 2.5 shows the existence of one solution with order $1/(2-\kappa)$. It can be shown that such solutions form an n-parametric family

$$x_j = A_j e^{i\varphi_j}(-t)^{\Omega}, \qquad y_j = iA_j e^{-i\varphi_j}(-t)^{\Omega}, \qquad j = 1, \dots, n. \tag{72}$$

Here, the parameters $\varphi_1, \dots, \varphi_n$ are real.

Remark 2.2. Similarly, and much simpler than Example 2.5, we can consider the case $\mathbf{p} > 0$, $\mathbf{q} = 0$. But the formulas (61) don't work there.

3. Vicinity of a periodic solution

3.1. Local coordinates

Let a real Hamiltonian system with $n+1$ degrees of freedom have a real 2π-periodic solution $\mathcal{M}$ and the Hamiltonian function is analytic in the neighborhood of the solution $\mathcal{M}$. According to Bruno[2], one can introduce such real local canonically conjugate coordinates $\boldsymbol{\xi}$, ψ and $\boldsymbol{\eta}$, ρ near the solution $\mathcal{M}$ that the solution $\mathcal{M}$ is given by equations

$$\boldsymbol{\xi} = \boldsymbol{\eta} = 0, \qquad \rho = 0, \qquad \psi = \psi_0 + t \tag{73}$$

and the Hamiltonian function has the form

$$\gamma = \Sigma \gamma_{\mathbf{pq}l}(\psi)\boldsymbol{\xi}^{\mathbf{p}}\boldsymbol{\eta}^{\mathbf{q}}\rho^l = \rho + \cdots, \tag{74}$$

where integer $\mathbf{p}, \mathbf{q} \geq 0$, integer $l \geq 0$, real analytic functions $\gamma_{\mathbf{pq}l}(\psi)$ have on ψ the period 2π and decompose into Fourier series.

Here, as in Section 2, there is a notion of Lyapunov stability, but at $n = 1$ the conditions for its existence coincide with those for formal stability.

Definition 3.1. Periodic solution (73) of a Hamiltonian system

$$\begin{aligned} \dot{\xi}_j &= \frac{\partial \gamma}{\partial \eta_j}, & \dot{\eta}_j &= -\frac{\partial \gamma}{\partial \xi_j}, \qquad j = 1, \dots, n, \\ \dot{\psi} &= \frac{\partial \gamma}{\partial \rho}, & \dot{\rho} &= -\frac{\partial \gamma}{\partial \psi} \end{aligned} \tag{75}$$

orbitally formally stable if there exists such a real power series on $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, ρ almost periodic on ψ

$$F = \sum F_{\mathbf{pq}l}(\psi)\boldsymbol{\xi}^{\mathbf{p}}\boldsymbol{\eta}^{\mathbf{q}}\rho^l \stackrel{\text{def}}{=} F_s(\boldsymbol{\xi}, \psi, \boldsymbol{\eta}, \rho) + \tilde{F}^{(s+1)}(\boldsymbol{\xi}, \psi, \boldsymbol{\eta}, \rho) \tag{76}$$

which may diverge, but is a formal sign-defined integral of the system (75).

In other words, all the coefficients of a power series

$$\sum_{j=1}^{n}\left(\frac{\partial F}{\partial \xi_j}\frac{\partial \gamma}{\partial \eta_j} - \frac{\partial F}{\partial \eta_j}\frac{\partial \gamma}{\partial \xi_j}\right) + \frac{\partial F}{\partial \psi}\frac{\partial \gamma}{\partial \rho} - \frac{\partial F}{\partial \rho}\frac{\partial \gamma}{\partial \psi} \tag{77}$$

must converge to zero, and they are homogeneous in $\boldsymbol{\zeta}, \sqrt{\rho}$ form $F_s(\boldsymbol{\xi}, \psi, \boldsymbol{\eta}, \rho) \geq 0$, with $F_s(\boldsymbol{\xi}, \psi, \boldsymbol{\eta}, \rho) = 0$ only when $\boldsymbol{\xi} = \boldsymbol{\eta} = 0$, $\rho = 0$.

Recall that a function $f(\psi)$ is

- periodic if it has a single frequency,

- conditionally (or quasi) periodic if it has a finite number of frequencies, and
- almost periodic if it has a countable number of frequencies.

In our case, there will be quasi-periodic functions $F_{\mathbf{pq}l}(\psi)$.

Definition 3.1 is similar to Definition 2.3, but one can also define formal orbital stability similar to Definition 2.4.

3.2. Normal form

When $\rho = 0$ and $\psi = t$, the quadratic on $\boldsymbol{\zeta}$ part γ_2 of the Hamiltonian (74) defines 2π-periodic, linear on $\boldsymbol{\zeta}$ system

$$\dot{\xi}_j = \frac{\partial \gamma_2}{\partial \eta_j}, \qquad \dot{\eta}_j = -\frac{\partial \gamma_2}{\partial \xi_j}, \qquad j = 1, \dots, n \tag{78}$$

Let $\nu_1, \dots, \nu_{2n}$ be the eigenvalues of its monodromy matrix, i.e., the substitution matrix of the fundamental matrix of solutions of the system (78) for period 2π. Let all $|\nu_j| = 1$ and $\nu_j \neq -1$. Assume

$$\alpha_j = \frac{\ln \nu_j}{2\pi}, \qquad \alpha_j \in \mathbb{R}, \alpha_j \in \left(-\frac{1}{2}, \frac{1}{2}\right), j = 1, \dots, 2n. \tag{79}$$

If the numeration is correct than $\alpha_{j+n} = -\alpha_j$, $j = 1, \dots, n$. Put $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$.

Condition $\boldsymbol{B_k^n}$. For all integer $\mathbf{p}$ with $\| \mathbf{p} \| \overset{\text{def}}{=} |p_1| + \dots + |p_n| \leq k$, the scalar products $\langle \mathbf{p}, \boldsymbol{\alpha} \rangle$ are not integers, i.e., the comparison $\langle \mathbf{p}, \boldsymbol{\alpha} \rangle \equiv 0 (\mathrm{mod} 1)$ has no solutions with such $\mathbf{p}$.

Theorem 3.1.[2,20] Given the condition B_2^n, there exists a complex formal reversible 2π-periodic on ψ and φ canonical coordinate transformation

$$\boldsymbol{\xi}, \psi, \boldsymbol{\eta}, \rho \leftrightarrow \mathbf{x}, \varphi, \mathbf{y}, r, \tag{80}$$

which brings the Hamiltonian γ to the normal form

$$g(\mathbf{x}, \varphi, \mathbf{y}, r) = r + i \sum_{j=1}^{n} \alpha_j x_j y_j + \sum g_{\mathbf{pq}lm} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}} r^l e^{im\varphi} \tag{81}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$, $0 \leq \mathbf{p}, \mathbf{q} \in \mathbb{Z}^n$, $l \geq 0$ and m are integers, all second sum terms of order $\mathbf{x}, \mathbf{y}, \sqrt{r}$ are above two and resonant, that is,

$$\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\alpha} \rangle + m = 0. \tag{82}$$

Let's put $r_j = x_j y_j$, $j = 1, \dots, n$; $\mathbf{r} = (r_1, \dots, r_n)$.

Corollary 3.1. If the condition B_4^n is satisfied, then the normal form (81), (82) has the form

$$g = r + i\langle \boldsymbol{\alpha}, \mathbf{r} \rangle + \langle C\mathbf{r}, \mathbf{r} \rangle + r\langle \boldsymbol{\delta}, \mathbf{r} \rangle + \varepsilon r^2 + \tilde{g}^{(5)} \tag{83}$$

where $\boldsymbol{\delta} = \text{const} \in \mathbb{C}^n$, $\varepsilon = \text{const} \in \mathbb{C}$ and $C \in \mathbb{C}^{n \times n}$.

Theorem 3.2.[20] The canonical transformation

$$x_j = u_j \mathbf{e}^{-\mathrm{i}\alpha_j \varphi}, \qquad y_j = v_j \mathbf{e}^{i\alpha_j \varphi}, \qquad j = 1, \dots, n, \tag{84}$$

$$r = s - i \sum_{j=1}^{n} \mathrm{Im} \lambda_j u_j v_j \tag{85}$$

leads the normal form of the Hamiltonian (81) to an autonomous power series

$$h(\mathbf{u},\mathbf{v},s) = s + \sum h_{\mathbf{pq}lm}\mathbf{u}^{\mathbf{p}}\mathbf{v}^{\mathbf{q}}s^{l} \tag{86}$$

corresponding to the second sum in (81).

Note that the returns from the variables $\mathbf{u},\mathbf{v},s$ to the original variables are given by formal power series on $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, ρ with quasi-periodic coefficients on ψ. Let us call the Hamiltonian (86) the reduced normal form.

The variable s is now the formal integral of the system

$$\dot{u}_j = \frac{\partial h}{\partial v_j}, \qquad \dot{v}_j = -\frac{\partial h}{\partial u_j}, \qquad j = 1,\dots,n \tag{87}$$

The orbital stability problem of the periodic solution $\mathcal{M}$ has now been reduced to the stability problem of the fixed point $\mathbf{u} = \mathbf{v} = 0$, $s = 0$ in the system (87).

Corollary 3.2. If the condition B_4^n is satisfied, then according to (83) and (85) the given normal form (86) is

$$\begin{aligned} h = \;& s + \langle C\mathbf{r},\mathbf{r}\rangle + (s - i\langle\boldsymbol{\alpha},\mathbf{r}\rangle)\langle\boldsymbol{\delta},\mathbf{r}\rangle + \varepsilon(s - i\langle\boldsymbol{\alpha},\mathbf{r}\rangle)^2 + \tilde{h}^{(5)} = \\ & s + \varepsilon s^2 + s\langle\boldsymbol{\delta},\mathbf{r}\rangle - \varepsilon s 2i\langle\boldsymbol{\alpha},\mathbf{r}\rangle + \langle C\mathbf{r},\mathbf{r}\rangle - i\langle\boldsymbol{\alpha},\mathbf{r}\rangle\langle\boldsymbol{\delta},\mathbf{r}\rangle - \varepsilon\langle\boldsymbol{\alpha},\mathbf{r}\rangle^2 + \tilde{h}^{(5)}. \end{aligned} \tag{88}$$

3.3. Real case

If the original Hamiltonian function γ is real under the real variables $\boldsymbol{\xi},\psi,\boldsymbol{\eta},\rho$, then in Theorem 3.1 the variables $\mathbf{x},\mathbf{y}$ are complex and the variables ψ,ρ and φ,r are real.

If the condition B_2^n is satisfied, then according to Bruno[2] the complex variables $\mathbf{x},\mathbf{y}$ are related to the real variables $\mathbf{X}$, $\mathbf{Y}$ by the formulae

$$x_j = \frac{X_j - Y_j}{\sqrt{2}}, \qquad y_j = \frac{X_j + Y_j}{\sqrt{2}}, \qquad j = 1,\dots,n \tag{89}$$

The complex variables x_j, y_j and their conjugate variables $\bar{x}_j, \bar{y}_j$ are related by the relations

$$\bar{x}_j = -iy_j, \qquad \bar{y}_j = -ix_j, \qquad j = 1,\dots,n, \qquad \bar{\varphi} = \varphi, \qquad \bar{r} = r. \tag{90}$$

With complex conjugation, the Hamiltonian (81) is preserved: $\bar{g}(\mathbf{x},\varphi,\mathbf{y},r) = g(\mathbf{x},\varphi,\mathbf{y},r)$. Indeed, $\bar{\imath}\alpha_j x_j y_j = \bar{\imath}\alpha_j \bar{x}_j \bar{y}_j = i\alpha_j x_j y_j$, and we can show that

$$\bar{g}_{\mathbf{pq}lm}(-)^{||\mathbf{p}+\mathbf{q}||} = g_{\mathbf{qp}l(-m)} \tag{91}$$

Note that according to (74) $\overline{ir_j} = -i\bar{x}_j\bar{y}_j = (-i)^3 x_j y_j = ix_j y_j = ir_j$, $j = 1,\dots,n$. Therefore, in (83), all μ_{jk} and ε are real, and all δ_j are purely imaginary. Assume $\boldsymbol{\delta} = 2i\boldsymbol{\Delta}$. According to (89)

$$r_j = x_j y_j = -\frac{1}{2i}\left(X_j^2 + Y_j^2\right) \overset{\text{def}}{=} \frac{i}{2}R_j, \qquad j = 1,\dots,n \tag{92}$$

Now (88) takes the form

$$h = s + \varepsilon s^2 - s\langle\boldsymbol{\Delta},\mathbf{R}\rangle + \varepsilon s\langle\boldsymbol{\alpha},\mathbf{R}\rangle - \frac{1}{4}\langle C\mathbf{R},\mathbf{R}\rangle - \frac{1}{2}\langle\boldsymbol{\alpha},\mathbf{R}\rangle\langle\boldsymbol{\Delta},\mathbf{R}\rangle + \frac{1}{4}\varepsilon\langle\boldsymbol{\alpha},\mathbf{R}\rangle^2 + \tilde{h}^{(5)} \tag{93}$$

All quantities here are real.

All integer vectors $\mathbf{q}$ that satisfy the comparison $\langle\boldsymbol{\alpha},\mathbf{q}\rangle \equiv 0(\mathrm{mod}1)$, form in $\mathbb{R}^n$ the lattice L. Let M be its linear shell and $Q=\{\mathbf{q}\geq 0,\mathbf{q}\neq 0\}$ is a non-negative orthant in $\mathbb{R}^n$ without the origin.

Theorem 3.3. If at $\rho=0$ the initial real system with Hamiltonian $\gamma(\boldsymbol{\xi},\psi,\boldsymbol{\eta},\rho)$ satisfies the condition B_4^n and in the entry (93)

$$\langle C\mathbf{q},\mathbf{q}\rangle + 2\langle\boldsymbol{\alpha},\mathbf{q}\rangle\langle\boldsymbol{\Delta},\mathbf{q}\rangle - \varepsilon\langle\boldsymbol{\alpha},\mathbf{q}\rangle^2 \neq 0 \tag{94}$$

for all $\mathbf{q}\in M\cap Q$, then the periodic solution (73) is formally orbitally stable.

Proof is similar to the proof of the theorem in “Formal stability of Hamiltonian systems”[7].

The given normal form (86) contains only the resonance terms satisfying Equation (82). And under the condition B_4^n it has the form (93). Therefore it has three types of real formal integrals:

1) $\langle\mathbf{q},\mathbf{R}\rangle$, where the vector $\mathbf{q}$ is orthogonal to the linear subspace M;

2) $H = h - s - \varepsilon s^2 = \varepsilon s\langle\boldsymbol{\alpha},\mathbf{R}\rangle + \frac{1}{4}\varepsilon\langle\boldsymbol{\alpha},\mathbf{R}\rangle^2 - s\langle\boldsymbol{\Delta}\mathbf{R}\rangle - \frac{1}{4}\langle C\mathbf{R},\mathbf{R}\rangle - \frac{1}{2}\langle\boldsymbol{\alpha},\mathbf{R}\rangle\langle\boldsymbol{\Delta},\mathbf{R}\rangle + \tilde{h}^{(5)}$;

3) s.

By condition (94) at $\mathbf{R}\in M$ the sum

$$\langle C\mathbf{R},\mathbf{R}\rangle + 2\langle\boldsymbol{\alpha},\mathbf{R}\rangle\langle\boldsymbol{\Delta},\mathbf{R}\rangle - \varepsilon\langle\boldsymbol{\alpha},\mathbf{R}\rangle^2$$

retains the sign and does not equal to zero. Let $\mathbf{R}\in M$ and

$$\begin{aligned}\mu_* &= \min\left|\frac{\langle C\mathbf{R},\mathbf{R}\rangle + 2\langle\boldsymbol{\alpha},\mathbf{R}\rangle\langle\boldsymbol{\Delta},\mathbf{R}\rangle - \varepsilon\langle\boldsymbol{\alpha},\mathbf{R}\rangle^2}{\langle\mathbf{R},\mathbf{R}\rangle}\right|,\\ \mu^* &= \max\left|\frac{\langle\boldsymbol{\Delta},\mathbf{R}\rangle - \varepsilon\langle\boldsymbol{\alpha},\mathbf{R}\rangle}{\sqrt{\langle\mathbf{R},\mathbf{R}\rangle}}\right|.\end{aligned}$$

According to the condition (94) we have $\mu_*>0$. Since s and $\langle\boldsymbol{\alpha},\mathbf{R}\rangle$ are integrals, the sum $H+As^2$ with any constant A is also an integral.

Consider the trinomial

$$-\frac{1}{4}\mu_*\lambda^2 - \mu^*\lambda s + As^2 \tag{95}$$

Its discriminant $D=(\mu^*)^2+\mu_*A$. If

$$\mu_*A<0 \text{ and } |\mu_*A|>(\mu^*)^2 \tag{96}$$

then the trinomial (95) has no real roots except $\lambda=s=0$.

Let $L_1,\dots,L_m$ be the basis of the orthogonal complement to the linear subspace M in $\mathbb{R}^n$. The sum $F=\sum_{j=1}^m \langle L_j,\mathbf{R}\rangle^4 + (H+As^2)^2 = F_8+\cdots$ is a formal integral as a polynomial of formal integrals. Let us show that with the number A with the property (96) the form

$$F_8 = \sum_{j=1}^{m}\langle L_j,\mathbf{R}\rangle^4 + \Big(\frac{1}{4}\varepsilon\langle\boldsymbol{\alpha},\mathbf{R}\rangle^2 - \frac{1}{4}\langle C\mathbf{R},\mathbf{R}\rangle$$
$$-\frac{1}{2}\langle\boldsymbol{\alpha},\mathbf{R}\rangle\langle\boldsymbol{\Delta},\mathbf{R}\rangle + s\varepsilon\langle\boldsymbol{\alpha},\mathbf{R}\rangle - s\langle\boldsymbol{\Delta},\mathbf{R}\rangle + As^2\Big)^2$$

is positive definite. Here in the right-hand side of the equality all the terms are greater than or equal to zero, for $R_j\geq 0$ for real X_j and Y_j and $\sum_{j=1}^m \langle L_j,\mathbf{R}\rangle^4 = 0$ only at $\mathbf{R}\in M$. But for such $\mathbf{R}$ at $\mathbf{R}\neq 0$ or $s\neq 0$

by the proved

$$\left(-\frac{1}{4}\langle C\mathbf{R},\mathbf{R}\rangle - s\langle\boldsymbol{\Delta},\mathbf{R}\rangle - \frac{1}{2}\langle\boldsymbol{\alpha},\mathbf{R}\rangle\langle\boldsymbol{\Delta},\mathbf{R}\rangle + As^2\right)^2 > 0.$$

So, F is the formal integral of the original system if in it $\mathbf{X},\varphi,\mathbf{Y},s$ is expressed through the old coordinates $\boldsymbol{\xi},\psi,\boldsymbol{\eta},\rho$. According to Definition 3.1 the initial system is formally orbitally stable. The proof is finished.

3.4. Cases $n = 1$ and $n = 2$

Under the condition B_4^n, the normal form of the Hamiltonian (83) in real coordinates is

$$g = r - \frac{1}{2}\langle\boldsymbol{\alpha},\mathbf{R}\rangle - \frac{1}{4}\langle C\mathbf{R},\mathbf{R}\rangle - r\langle\boldsymbol{\Delta},\mathbf{R}\rangle + \varepsilon r^2 + \cdots$$

Here, there are linear and quadratic parts on $\mathbf{R}, r$, and the situation is like a normal form of the Hamiltonian with $n+1$ degrees of freedom in the neighborhood of the fixed point.

Therefore, according to Markeev[21,22], for $n = 1$ and $n = 2$ respectively formulated without proof formal stability conditions similar to the Markeev 2 condition from Subsection 2.3. Namely:

Markeev's condition 3. A system of two equations

$$q_0 - \frac{1}{2}\langle\boldsymbol{\alpha},\mathbf{q}\rangle = 0,$$
$$-\frac{1}{4}\langle C\mathbf{q},\mathbf{q}\rangle - q_0\langle\boldsymbol{\Delta},\mathbf{q}\rangle + \varepsilon q_0^2 = 0$$

has no solution $\mathbf{q} \geq 0$, $|q_0| + \| \mathbf{q} \| \neq 0$, i.e., equation

$$\langle C\mathbf{q},\mathbf{q}\rangle + 2\langle\boldsymbol{\alpha},\mathbf{q}\rangle\langle\boldsymbol{\Delta},\mathbf{q}\rangle - \varepsilon\langle\boldsymbol{\alpha},\mathbf{q}\rangle^2 = 0, \quad (*)$$

has no solution $\mathbf{q} \geq 0$.

It differs from the condition (94) of Theorem 3.3 and is easier to check.

3.5. Scattering order of solution

Definition 3.2. Let the solution

$$\boldsymbol{\xi}(t), \varphi(t), \boldsymbol{\eta}(t), \rho(t) \tag{97}$$

tends to a periodic solution (73) at $t \to -\infty$. On the solution (97) the *order of the expansion*

$$\Delta = \min\{\delta(||\boldsymbol{\xi}||), \delta(||\boldsymbol{\eta}||), \delta(\sqrt{|\rho|})\} \tag{98}$$

Definition 3.3. *Scattering order of solution* to the system (75) from its periodic solution (73) $\tilde{\Delta}$ is the lower bound of the Δ scatter order over all solutions (97) that tend to the periodic solution (73) at $t \to -\infty$.

Here, as in Section 2.6, we estimate the order of dispersion of solutions from a periodic solution in the absence of formal stability.

Conjecture 3.1. Let the condition B_2^n be satisfied and $\kappa = min \parallel \mathbf{p} + \mathbf{q} \parallel > 2$ by integer solutions $\mathbf{p} \geq 0$, $\mathbf{q} \geq 0$, m of the equation $\langle\boldsymbol{\alpha},\mathbf{p} - \mathbf{q}\rangle + m = 0$, then the order of dispersion of the solutions of the system (75) from the periodic solution, denoted by $\tilde{\Delta} \geq (2-\kappa)^{-1}$.

An example similar to 2.5 and using Equation (91) is recommended for the reader to construct.

Author contributions

Conceptualization, ADB; methodology, ADB; software, ABB; validation, ADB and ABBB; writing—original draft preparation, ADB and ABB; writing—review and editing, ADB and ABB; visualization, ABB. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no conflict of interest.

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