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ORIGINAL RESEARCH ARTICLE

Faber polynomials estimates for bi-univalent functions of complex order involving *q*-derivative

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ABSTRACT

In this paper, we define a new subclass of bi-univalent functions of complex order $\sum_q(\tau,\zeta;\varphi)$ which is defined by subordination in the open unit disc D by using $\nabla_q F(\vartheta)$ operator. Furthermore, using the Faber polynomial expansions, we get upper bounds for the coefficients of function belonging to this class. It is known that the calculus without the notion of limits is called q-calculus which has influenced many scientific fields due to its important applications. The generalization of derivative in q-calculus that is q-derivative was defined and studied by Jackson. A function $F \in A$ is said to be bi-univalent in D if both F and F^{-1} are univalent in D. The class consisting of bi-univalent functions is denoted by σ. The Faber polynomials play an important role in various areas of mathematical sciences, especially in geometric function theory. The purpose of our study is to obtain bounds for the general coefficients $|a_n|(n \geq 3)$ by using Faber polynomial expansion under certain conditions for analytic bi-univalent functions in subclass $\sum_q(\tau,\zeta;\phi)$ and also, we obtain improvements on the bounds for the first two coefficients $|a_2|$ and $|a_3|$ of functions in this subclass. In certain cases, our estimates improve some of those existing coefficient bounds.

Keywords: analytic functions; bi-univalent functions; coefficient bounds; subordination; *q*-derivative; Faber polynomials

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1. Introduction

Let A be the class of functions

$$\mathrm{F}(\vartheta) = \vartheta + \sum_{\varepsilon=2}^{\infty} a_\varepsilon \vartheta^\varepsilon, \tag{1}$$

defined in $D = \{\vartheta \in C \ : |\vartheta| < 1\}$ normalized by the conditions $F(0) = F'(0) - 1 = 0$ for every $\vartheta \in D$ and S be the subclass of A consisting of univalent functions in D. For every $F \in S$ there exists an inverse function F^{-1} which is defined in some neighborhood of the origin, and satisfying the conditions,

$$F^{-1}(F(\vartheta)) = \vartheta, (\vartheta \in D)$$

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and,

$$F^{-1}(F(\omega)) = \omega, (\vartheta < r_0(F); r_0(F) \geq \frac{1}{4},$$

where,

$$g(\omega) = \mathrm{F}^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 + -(5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \ldots = \omega + \sum_{\varepsilon=2}^{\infty} A_\varepsilon \omega^\varepsilon. \quad (2)$$

If both F and F^{-1} are univalent in D, then $F \in A$ is called bi-univalent in D and the class of these functions is denoted by σ. For there are many studies in this class[1–6].

Faber[7] introduced a polynomial which bears his name and is very important role in geometric function theory.

By using the expansion of this polynomial for $F \in S$, the coefficients of its inverse $g = F^{-1}$ may be expressed as[8,9]

$$g(\omega) = \mathrm{F}^{-1}(\omega) = \omega + \sum_{\varepsilon=2}^{\infty} \frac{1}{\varepsilon} \chi_{\varepsilon-1}^{-\varepsilon}(a_2, a_3, \ldots, a_\varepsilon)\omega^\varepsilon, \quad (3)$$

where,

$$\begin{aligned}
\chi_{\varepsilon-1}^{-\varepsilon} &= \frac{(-\varepsilon)!}{(-2\varepsilon+1)!\,(\varepsilon-1)!} a_2^{\varepsilon-1} + \frac{(-\varepsilon)!}{(2(-\varepsilon+1))!\,(\varepsilon-3)!} a_2^{\varepsilon-3} a_3 \\
&= \frac{(-\varepsilon)!}{(-2\varepsilon+3)!\,(\varepsilon-4)!} a_2^{\varepsilon-4} a_4 + \frac{(-\varepsilon)!}{(2(-\varepsilon+2))!\,(\varepsilon-5)!} a_2^{\varepsilon-5} \\
&\times [a_5 + (-\varepsilon+2)a_3^2] + \frac{(-\varepsilon)!}{(-2\varepsilon+5))!\,(\varepsilon-6)!} a_2^{\varepsilon-6} \times [a_6 + (-2\varepsilon+5)a_3a_4] \\
&+ \sum_{j\geq 7}^{\infty} a_2^{\varepsilon-j} V_j,
\end{aligned}$$

such that V_j with $7 \leq j \leq \varepsilon$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_\varepsilon$[9]. The first three terms of $\chi_{\varepsilon-1}^{-\varepsilon}$ are

$$\chi_1^{-2} = -2a_2,\ \chi_2^{-3} = 3(2a_2^2 - a_3),\ \chi_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

In general, for any $p \in \vartheta = \{0, \pm1, \pm2, \ldots\}$, an expansion of χ_ε^p is

$$\chi_\varepsilon^p = pa_{\varepsilon+1} + \frac{p(p-1)}{2} H_\varepsilon^2 + \frac{p!}{(p-3)!\,3!} H_\varepsilon^3 + \ldots + \frac{p!}{(P-\varepsilon)!\,\varepsilon!} H_\varepsilon^\varepsilon,$$

where $H_\varepsilon^p = H_\varepsilon^p(a_2, a_3, \ldots)$ and by Jahangiri et al.[10], (see for details[8,9,11,12,13])

$$H_\varepsilon^m(a_2, a_3, \ldots, a_{\varepsilon+1}) = \sum_{\varepsilon=0}^{\infty} \frac{m!\,(a_2)^{\mu_1} \ldots (a_{\varepsilon+1})^{\mu_\varepsilon}}{\mu_1! \ldots \mu_\varepsilon!}, \quad (4)$$

where the sum is taken $\forall \mu_1, \ldots, \mu_\varepsilon \in N = \{1, 2, \ldots\}$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + \ldots + \mu_\varepsilon = m, \\ \mu_1 + 2\mu_2 + \ldots + k\mu_\varepsilon = \varepsilon. \end{cases}$$

Note that $H_\varepsilon^\varepsilon(a_2, a_3, \ldots, a_{\varepsilon+1}) = a_2^\varepsilon$.

In the rest of this paper, assume that φ is an analytic function with positive real part in D, satisfying $\varphi(0) = 1,\ \varphi'(0) > 0$ and $\varphi(D)$ is symmetric w. r. to the real axis and has the expansion,

$$\varphi(\vartheta) = 1 + \psi_1\vartheta + \psi_2\vartheta^2 + \psi_3\vartheta^3 + \ldots \quad (\psi_1 > 0).$$

Let $u(\vartheta)$ and $v(\vartheta)$ be analytic in D with $u(0) = v(0) = 0,\ |u(\vartheta)| < 1,\ |v(\vartheta)| < 1$, and

$$u(\vartheta)=\vartheta(p_1+\sum_{\varepsilon=2}^{\infty}p_\varepsilon\vartheta^{\varepsilon-1}) \text{ and } v(\vartheta)=\vartheta(q_1+\sum_{\varepsilon=2}^{\infty}q_\varepsilon\vartheta^{\varepsilon-1}) \quad (\vartheta\in \mathrm{D}). \tag{5}$$

Then[14],

$$p_1\le 1,\qquad p_\varepsilon\le 1-p_1,\qquad q_1\le 1,\qquad q_\varepsilon\le 1-q_1,\qquad (\varepsilon\in \mathcal{N}\backslash\{1\}). \tag{6}$$

Jackson[15] $q-$ derivative, $0<q<1$, was defined by Annby and Mansour and other researchers[16–21]:

$$\nabla_q F(\vartheta)=\begin{cases}\frac{F(\vartheta)-F(q\vartheta)}{(1-q)\vartheta}, & \vartheta\neq 0\\ F'(0), & \vartheta=0\end{cases},$$

that is,

$$\nabla_q F(\vartheta)=1+\sum_{\varepsilon=2}^{\infty}[\varepsilon]_q a_\varepsilon\vartheta^{\varepsilon-1}, \tag{7}$$

where,

$$[j]_q=\frac{1-q^j}{1-q},\qquad [0]_q=0. \tag{8}$$

As $q\to 1^-$, $[j]_q=j$ and $\nabla_q F(\vartheta)=F'(\vartheta)$.

Definition 1: For F, g, analytic in D, F is subordinate to g in D written $F\prec g$, if $\exists\ \Omega(\vartheta)$, analytic in D, with $\Omega(0)=0$ and $|\Omega(\vartheta)|<1$ $(\vartheta\in D)$ such that $F(\vartheta)=g(\Omega(\vartheta))$ $(\vartheta\in D)$[22,23].

Definition 2: For $\tau\in C^*=C?\{0\}$, $0\le\zeta\le 1$, $0<q<1$ and $F\in\sigma$, $F\in\sum_q(\tau,\zeta;\varphi)$ if for all $\vartheta,\omega\in D$:

$$1+\frac{1}{\tau}\left[\nabla_q(F(\vartheta))+\zeta\vartheta\nabla_q(\nabla_q F(\vartheta))-1\right]\prec\varphi(\vartheta), \tag{9}$$

and

$$1+\frac{1}{\tau}\left[\nabla_q(g(\omega))+\zeta\omega\nabla_q(\nabla_q g(\omega))-1\right]\prec\varphi(\omega), \tag{10}$$

where $g(\omega)=F^{-1}(\omega)$.

Note that:

1) $\sum_q(1,\zeta;\phi)=\sum_q(\zeta;\phi)$;

2) $\sum_q((1-\alpha)e^{-i\theta}\cos\theta,\zeta;\phi)=\sum_q(\zeta,\alpha,\theta;\phi)$, $(0\le\alpha<1,\ |\theta|<\frac{\pi}{2})$, where

$$=\begin{cases}F\in\sigma\ :\ \dfrac{e^{i\theta}\left[\nabla_q(F(\vartheta))+\zeta\vartheta\nabla_q(\nabla_q F(\vartheta))\right]-(\alpha\cos\theta+i\sin\theta)}{(1-\alpha)\cos\theta}\prec\varphi(\vartheta)\\ g\in\sigma\ :\ \dfrac{e^{i\theta}\left[\nabla_q(g(\omega))+\zeta\omega\nabla_q(\nabla_q g(\omega))\right]-(\alpha\cos\theta+i\sin\theta)}{(1-\alpha)\cos\theta}\prec\varphi(\omega)\end{cases};$$

3) $\lim_{q\to 1^-}\sum_q(\tau,\zeta;\phi)=\sum(\tau,\zeta;\phi)$[24];

4) $\lim_{q\to 1^-}\sum_q(1,\zeta;\phi)=\sum(\zeta;\phi)$[25];

5) $\lim_{q\to 1^-}\sum_q((1-\alpha)e^{-i\theta}\cos\theta,\zeta;\frac{1+\vartheta}{1-\vartheta})=\sum(\zeta,\alpha,\theta;\frac{1+\vartheta}{1-\vartheta})$, $(0\le\alpha<1\ |\theta|<\frac{\pi}{2})$,

$$= \begin{cases} F \in \sigma \ : \ \dfrac{e^{i\theta}[F'(\vartheta) + \zeta\vartheta F''(\vartheta)] - (\alpha\cos\theta + i\sin\theta)}{(1-\alpha)\cos\theta} \prec \dfrac{1+\vartheta}{1-\vartheta} \\ g \in \sigma \ : \ \dfrac{e^{i\theta}[g'(\omega) + \zeta\omega g''(\omega)] - (\alpha\cos\theta + i\sin\theta)}{(1-\alpha)\cos\theta} \prec \dfrac{1+\omega}{1-\omega} \end{cases}.$$

2. Main results

We assume that $\tau \in C^*,\ 0 < q < 1,\ 0 \le \zeta \le 1$ and $F(\vartheta) \in \sigma$.

In this section we obtain some inequalities for the function class $\sum_q(\tau, \zeta; \phi)$.

Theorem 1: Let $\mathcal{F} \in \sum_q(\tau, \zeta; \phi)$. If $a_\varepsilon = 0$ for $2 \le \varepsilon \le \epsilon - 1$ then,

$$|a_\epsilon| \le \frac{\psi_1 |\tau|}{(1 + \zeta[\epsilon-1]_q)\, [\epsilon]_q} \ \ (\epsilon \ge 3) \tag{11}$$

Proof: For functions $\nabla_q \mathcal{F}(\vartheta)$ given by Equation (7) and g = F^{-1} we have:

$$\begin{aligned} & 1 + \frac{1}{\tau}\left[\nabla_q(\mathcal{F}(\vartheta)) + \zeta\vartheta\nabla_q(\nabla_q\mathcal{F}(\vartheta)) - 1\right] \\ = \ & 1 + \frac{1}{\tau}\sum_{\epsilon=2}^{\infty}(1 + \zeta[\epsilon-1]_q)\,[\epsilon]_q a_\epsilon \vartheta^{\epsilon-1}, \end{aligned} \tag{12}$$

$$\begin{aligned} & 1 + \frac{1}{\tau}\left[\nabla_q(g(\omega)) + \zeta\omega\nabla_q(\nabla_q g(\omega)) - 1\right] \\ = \ & 1 + \frac{1}{\tau}\sum_{\epsilon=2}^{\infty}(1 + \zeta[\epsilon-1]_q)\,[\epsilon]_q A_\epsilon \omega^{\epsilon-1}. \end{aligned} \tag{13}$$

Using Equation (3), we have:

$$\begin{aligned} & 1 + \frac{1}{\tau}\left[\nabla_q(g(\omega)) + \zeta\omega\nabla_q(\nabla_q g(\omega)) - 1\right] \\ = \ & 1 + \frac{1}{\tau}\sum_{\epsilon=2}^{\infty}(1 + \zeta[\epsilon-1]_q)\,[\epsilon]_q \frac{1}{\epsilon}\chi_{\epsilon-1}^{-\epsilon}(a_2, a_3, \ldots, a_\epsilon)\omega^{\epsilon-1}. \end{aligned} \tag{14}$$

Considering Equations (9) and (10), there are two Schwarz functions $u,\ v : \mathbb{D} \to \mathbb{D}$ with $u(0) = v(0) = 0$, which are given by Equation (5), so that,

$$1 + \frac{1}{\tau}\left[\nabla_q(F(\vartheta)) + \zeta\vartheta\nabla_q(\nabla_q F(\vartheta)) - 1\right] = \varphi(u(\vartheta)), \tag{15}$$

$$1 + \frac{1}{\tau}\left[\nabla_q(g(\omega)) + \zeta\omega\nabla_q(\nabla_q g(\omega)) - 1\right] = \varphi(v(\omega)). \tag{16}$$

Also, by Equation (4) we get:

$$\begin{aligned} \phi(u(\vartheta)) &= 1 + \psi_1 p_1 \vartheta + (\psi_1 p_2 + \psi_2 p_1^2)\vartheta^2 + \ldots \\ &= 1 + \sum_{\epsilon=1}^{\infty}\sum_{\varepsilon=1}^{\epsilon}\psi_\varepsilon D_\epsilon^\varepsilon(p_1, p_2, \ldots, p_\epsilon)\vartheta^\epsilon \quad (\vartheta \in \mathbb{D}), \end{aligned} \tag{17}$$

and,

$$\begin{aligned}\phi(v(\omega)) &= 1+\psi_1 q_1\omega+(\psi_1 q_2+\psi_2 q_1^2)\omega^2+\dots \\ &= 1+\sum_{\epsilon=1}^{\infty}\sum_{\varepsilon=1}^{\epsilon}\psi_\varepsilon D_\epsilon^\varepsilon(q_1,q_2,\dots,q_\epsilon)\omega^\epsilon \quad (\omega\in\mathbb{D}).\end{aligned} \tag{18}$$

Comparing the coefficients of Equations (12), (15) and (17), we get:

$$\frac{1}{\tau}(1+\zeta[\varepsilon-1]_q)[\varepsilon]_q a_\varepsilon=\sum_{\iota=1}^{\varepsilon-1}\psi_\iota H_{\varepsilon-1}^{\iota}(p_1,p_2,\dots,p_{\varepsilon-1})\ (\varepsilon\geq 2). \tag{19}$$

Similarly, from Equations (14), (16) and (18), we get:

$$\frac{1}{\tau}\left(1+\zeta[\epsilon-1]_q\right)[\epsilon]_q\frac{1}{\epsilon}\chi_{\epsilon-1}^{-\epsilon}(a_2,a_3,\dots,a_\epsilon)=\sum_{\varepsilon=1}^{\epsilon-1}\psi_\varepsilon D_{\epsilon-1}^{\varepsilon}(q_1,q_2,\dots,q_{\epsilon-1}) \qquad (\epsilon\geq 2). \tag{20}$$

Now, from $a_\varepsilon=0$ for $2\leq\varepsilon\leq\epsilon-1$, we have $A_\epsilon=-a_\epsilon$ and the Equations (19) and (20) yield

$$\begin{aligned}\left(1+\zeta[\epsilon-1]_q\right)[\epsilon]_q a_\epsilon &= \tau\psi_1 p_{\epsilon-1}, \\ -\left(1+\zeta[\epsilon-1]_q\right)[\epsilon]_q a_\epsilon &= \tau\psi_1 q_{\epsilon-1}.\end{aligned} \tag{21}$$

Taking the modulus of each of the two equations in Equation (21) and using Equation (6), we obtain Equation (11).

Corollary 1: For $\varphi(\vartheta)=(\frac{1+\vartheta}{1-\vartheta})^\alpha\ (0<\alpha\leq 1)$, let $\mathcal{F}\in\sum_q(\tau,\zeta;\phi)$ then

$$|a_\epsilon|\leq\frac{2\alpha|\tau|}{\left(1+\zeta[\epsilon-1]_q\right)[\epsilon]_q} \qquad (\epsilon\geq 3). \tag{22}$$

Corollary 2: For $\varphi(\vartheta)=\frac{1+(1-2\beta)\vartheta}{1-\vartheta}\ (0\leq\beta<1)$, let $\mathcal{F}\in\sum_q(\tau,\zeta;\phi)$ then

$$|a_\epsilon|\leq\frac{2|\tau|(1-\beta)}{\left(1+\zeta[\epsilon-1]_q\right)[\epsilon]_q} \qquad (\epsilon\geq 3). \tag{23}$$

Remark 1: For $\tau=1,\ q\to 1^-$ in Corollary 2, we get results obtained by Srivastava et al.[13], for all $0\leq\zeta\leq 1$.

Theorem 2: Let $\mathcal{F}\in\sum_q(\tau,\zeta;\phi)$. Then,

$$|a_2|\leq\frac{\psi_1\sqrt{\psi_1}|\tau|}{\sqrt{\psi_1[2]_q^2(1+\zeta)^2+\left|\tau[3]_q(1+\zeta[2]_q)\psi_1^2-[2]_q^2(1+\zeta)^2\psi_2\right|}}, \tag{24}$$

$$|a_3|\leq\min\{\mathcal{K}(\zeta),\mathcal{L}(\zeta)\}, \tag{25}$$

where,

$$\mathcal{L}(\zeta)=\begin{cases}\dfrac{\psi_1|\tau|}{[3]_q(1+\zeta[2]_q)}\times\dfrac{[3]_q(1+\zeta[2]_q)|\tau|\psi_1^2+\left|[3]_q(1+\zeta[2]_q)\tau\psi_1^2-[2]_q^2(1+\zeta)^2\psi_2\right|}{[2]_q^2(1+\zeta)^2\psi_1+\left|[3]_q(1+\zeta[2]_q)\tau\psi_1^2-[2]_q^2(1+\zeta)^2\psi_2\right|} & ,\psi_1\geq\dfrac{[2]_q^2(1+\zeta)^2}{[3]_q(1+\zeta[2]_q)|\tau|} \\ \dfrac{\psi_1|\tau|}{[3]_q(1+\zeta[2]_q)} & ,0\leq\psi_1\leq\dfrac{[2]_q^2(1+\zeta)^2}{[3]_q(1+\zeta[2]_q)|\tau|}\end{cases}, \tag{26}$$

and

$$\mathcal{K}(\zeta)=\begin{cases}\frac{|\psi_2|\tau}{[3]_q(1+\zeta[2]_q)} & , |\psi_2|>\psi_1 \\ \frac{\psi_1\tau}{[3]_q(1+\zeta[2]_q)} & , |\psi_2|\leq\psi_1\end{cases}. \tag{27}$$

Proof: If we set $\varepsilon = 2$ and $\varepsilon = 3$ in Equations (19) and (20), respectively, we have

$$\frac{1}{\tau}[2]_q(1+\zeta)a_2=\psi_1 p_1, \tag{28}$$

$$\frac{1}{\tau}[3]_q\big(1+\zeta[2]_q\big)a_3=\psi_1 p_2+\psi_2 p_1^2, \tag{29}$$

$$-\frac{1}{\tau}[2]_q(1+\zeta)a_2=\psi_1 q_1, \tag{30}$$

and

$$\frac{1}{\tau}[3]_q\big(1+\zeta[2]_q\big)(2a_2^2-a_3)=\psi_1 q_2+\psi_2 q_1^2. \tag{31}$$

From Equations (28) and (30), we obtain:

$$p_1=-q_1. \tag{32}$$

Adding Equations (29) and (31), and using Equation (32), we have:

$$\frac{2}{\tau}[3]_q\big(1+\zeta[2]_q\big)a_2^2-2p_1^2\psi_2=\psi_1(p_2+q_2). \tag{33}$$

From Equation (28), we get:

$$\left[2\tau[3]_q\psi_1^2\left(1+\zeta[2]_q\right)-2[2]_q^2\left(1+\zeta\right)^2\psi_2\right]a_2^2=\tau^2\psi_1^3(p_2+q_2). \tag{34}$$

By Equations (6), (28) and (32), we obtain:

$$\begin{aligned}&\left|2\tau[3]_q\psi_1^2\left(1+\zeta[2]_q\right)-2[2]_q^2\left(1+\zeta\right)^2\psi_2\right|\left|a_2\right|^2\\ \leq\;& |\tau|^2\psi_1^3(|p_2|+|q_2|)\\ \leq\;& 2|\tau|^2\psi_1^3(1-|p_1|^2)\\ =\;& 2|\tau|^2\psi_1^3-2[2]_q^2\left(1+\zeta\right)^2\psi_1\left|a_2\right|^2.\end{aligned} \tag{35}$$

Consequently,

$$|a_2|^2\leq\frac{|\tau|^2\psi_1^3}{[2]_q^2\left(1+\zeta\right)^2\psi_1+\left|\tau[3]_q\psi_1^2\left(1+\zeta[2]_q\right)-[2]_q^2\left(1+\zeta\right)^2\psi_2\right|}.$$

So, we obtain the bound on $|a_2|$ in Equation (24).

Next, in order to find the bound on the coefficient $|a_3|$, by subtracting Equation (31) from Equation (29), and using Equation (32), we get:

$$\frac{2}{\tau}[3]_q\big(1+\zeta[2]_q\big)a_2^2-\frac{2}{\tau}[3]_q\big(1+\zeta[2]_q\big)a_3=\psi_1(q_2-p_2). \tag{36}$$

Using Equation (6), we have:

$$\begin{aligned}2[3]_q\left(1+\zeta[2]_q\right)|a_3| &\leq 2[3]_q\left(1+\zeta[2]_q\right)|a_2|^2+2|\tau|\psi_1(|p_2|+|q_2|)\\ &\leq 2[3]_q\left(1+\zeta[2]_q\right)|a_2|^2+2|\tau|\psi_1(1-|p_1|^2).\end{aligned} \tag{37}$$

From Equation (28), we get:

$$[3]_q\left(1+\zeta[2]_q\right)|\tau|\,\psi_1\,|a_3| \quad\leq\quad |\tau|^2\,\psi_1^2+\left[|\tau|\,[3]_q\psi_1\left(1+\zeta[2]_q\right)-[2]_q^2\left(1+\zeta\right)^2\right]|a_2|^2. \tag{38}$$

On the other hand, from Equation (29), we have:

$$[3]_q\left(1+\zeta[2]_q\right)|a_3|\leq\tau\left[\psi_1(1-|p_1|^2)+|\psi_2|\,|p_1|^2\right].$$

Consequently,

$$|a_3|\leq\begin{cases}\frac{|\psi_2|\tau}{[3]_q(1+\zeta[2]_q)} & ,\ |\psi_2|>\psi_1\\ \frac{\psi_1\tau}{[3]_q(1+\zeta[2]_q)} & ,\ |\psi_2|\leq\psi_1\end{cases}. \tag{39}$$

Hence, from Equations (38) and (39), we obtain Equation (25).

By letting $\zeta=0,\ \tau=1,$ we have:

Corollary 3: Let $\mathcal{F}\in\sum_q(1,0;\phi)$. Then

$$|a_3|\leq\min\left\{\mathcal{K}(0),\mathcal{L}(0)\right\}, \tag{40}$$

where,

$$\mathcal{L}(0)=\begin{cases}\frac{\psi_1}{[3]_q}\times\frac{[3]_q\psi_1^2+\left|[3]_q\psi_1^2-[2]_q^2\psi_2\right|}{[2]_q^2\psi_1+\left|[3]_q\psi_1^2-[2]_q^2\psi_2\right|} & ,\psi_1\geq\frac{[2]_q^2}{[3]_q}\\ \frac{\psi_1}{[3]_q} & ,0\leq\psi_1\leq\frac{[2]_q^2}{[3]_q}\end{cases}, \tag{41}$$

and,

$$\mathcal{K}(0)=\begin{cases}\frac{|\psi_2|}{[3]_q} & ,\ |\psi_2|>\psi_1\\ \frac{\psi_1}{[3]_q} & ,\ |\psi_2|\leq\psi_1\end{cases}. \tag{42}$$

3. Future work

The authors suggest to fined upper bounds for the coefficients of function class $\sum_{\lambda,q}^{m}(\tau,\zeta;\phi)$ for all $\vartheta,\omega\in D$:

$$1+\frac{1}{\tau}\left[\nabla_{\lambda,q}^{m}(F(\vartheta))+\zeta\vartheta\nabla_q(\nabla_{\lambda,q}^{m}F(\vartheta))-1\right]\prec\varphi(\vartheta), \tag{43}$$

and,

$$1+\frac{1}{\tau}\left[\nabla_{\lambda,q}^{m}(g(\omega))+\zeta\omega\nabla_q(\nabla_{\lambda,q}^{m}g(\omega))-1\right]\prec\varphi(\omega), \tag{44}$$

where,

$$\nabla_{\lambda,q}^{m}(F(\vartheta))=\vartheta+\sum_{k=2}^{\infty}[1+\lambda([k]_q-1)]^m a_k\vartheta^k,\lambda\geq0,m\in N_0=N\cup\{0\}, \tag{45}$$

is the $q-$ Al-Oboudi operator is defined by Aouf et al.[26].

4. Conclusions

Throughout the paper, we defined a new subclass of bi-univalent functions of complex order $\sum_q(\tau,\zeta;\phi)$ by using $\nabla_q F(\vartheta)$ operator. Furthermore, using the Faber polynomial expansions, we find the initial coefficient bounds for this function class. This paper generalized many results.

Author contributions

Conceptualization, AOM and SMM; methodology, AOM and SMM; validation, ZMS, AOM and SMM; formal analysis, ZMS, AOM and SMM; investigation, ZMS, AOM and SMM; resources, AOM and SMM; data curation, ZMS; writing—original draft preparation, ZMS; writing—review and editing, AOM and SMM; supervision, AOM and SMM. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no conflict of interest.

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