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ORIGINAL RESEARCH ARTICLE

Pricing of European call options using generalized Wishart processes

Joab Onyango Odhiambo

Department of Mathematics, Meru University of Science and Technology, Meru 972-60200, Kenya; joabodhiambo2030@gmail.com

ABSTRACT

This study explores a multiple-security, high-risk pricing model where the implied volatility has been portrayed through Generalized Wishart affine processes. The presence of dual dependency matrices distinctively characterizes this multifaceted model. These matrices encapsulate the relationship between the generalized Wishart processes and the evolving dynamics of several securities. The adaptability of the proposed model makes it a perfect fit for high-frequency market data, whether dealing with either long or short-term maturities of calls. The main objective paper is on its derivation and addressing the call option pricing problem within the context of the volatility mode using generalized Wishart stochastic. A combination of Fourier transforms techniques and perturbation methods are utilized, mainly focusing on pricing European call options. The model proposed in this study is theoretical and practical, showcasing the strong potential for real-world applications within the financial derivative market.

Keywords: generalized Wishart processes; perturbation methods; Fourier Transforms; infinitesimal generator

1. Introduction

The conventional Black and Scholes model, established in 1973, has certain limitations, notably its failure to account for the implied security volatility when pricing financial derivative instruments that vary by exercise and maturity dynamics. It makes it less adaptable to mirror specific market conditions recorded in financial derivative prices. This shortcoming led to the discovery of the Heston pricing model in 1993. The Heston model quickly gained popularity and has been extensively utilized in financial derivatives markets due to its superior adaptability, insightful financial parameters clarification, and the analytical tractability it offers, as it falls under the category of affine processes. These traits facilitate the calculation of the call value for a European Option by reversing the Fourier transforms and creating a specific closed-form solution for the characteristic function of log-prices of securities. The Heston pricing model provides a more nuanced approach, allowing for better adaptation to dynamic market conditions and greater accuracy in financial modelling.

Despite its popularity, the Heston model has documented limitations. Numerous studies, including those by Ahdida and Alfonsi[1], Christoffersen et al.[2], Odhiambo[3], Gourieroux[4], Heston[5], Shreve[6], have highlighted a key flaw: the model doesn't accurately produce the true term structure of volatility movements. It suggests that the Heston model's implied volatility surface is too flat to mirror certainty accurately. Normally, the implied volatility curve is locally convex for short maturities and tends to linearize for longer maturities.

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It indicates that the Heston model needs help to reflect financial market data when pricing derivative products accurately.

Several studies, including those by Bjork[7], Bru[8], Duffie et al.[9], Odhiambo et al.[10], Kang and Kang[11], Chandru et al.[12], and Odhiambo et al.[13] recommended addressing these deficiencies by expanding upon the Heston model in two ways: incorporating jumps in the security dynamic or volatility, and exploring the implied volatility multifactor nature. It's widely recognized that a multifactor approach can better handle the pricing problem of financial derivatives and volatility skew, prompting the adoption of the Wishart multidimensional stochastic volatility model, a stochastic matrix-defined process.

Introduced by Bru in 1991, the Wishart process is one positive semi-definite matrix-valued generalization of a Bessel function process that encapsulates multiple chi-squared distributions were pioneers in the application of the Wishart stochastic implied volatility model to the realm of finance Odhiambo et al.[13]. Since then, this model has gained considerable traction and is now extensively employed within derivatives markets. Its matrix specification nature offers flexibility, making it a more robust choice for modelling complex market dynamics, as shown in Odhiambo et al.[14] to help in modelling.

This study proposes to address the valuation of the European call option under the framework of generalized Wishart variance processes. Expanding the Heston model to encompass the multifactor nature of implied volatility modelling, with two dependence matrices for a particular security, the security dynamics are assumed to rely on two separate Wishart volatility processes, termed "double Wishart volatility processes". The matrix specifications inherent to the model make it possible to encapsulate stylized facts observed in financial markets by Naryongo et al.[15] and Odhiambo[16]. This adaptation allows for the effective pricing of options, regardless of whether the maturity is short or long, at any given level of volatility.

The strategy for addressing the pricing issue associated with European call options employs transform methods, as illustrated by Benabid et al.[17]. Additionally, the Fast Fourier Transform technique, as Benabid et al.[17] proposed, plays a significant role in this process. These techniques are used in conjunction with perturbation methodologies. Considering that ordinary differential equations doesn't permit a closed-form solution due to the non-commutative nature of the matrix multiplications involved, these methodologies are utilized to ascertain an approximation for the valuation of a specific European call option.

Filipovic and Mayerhofer[18] devised a numerical methodology to solve parabolic issues dominated by boundary and interior layers, specifically emphasising discontinuous convection coefficient and source terms. Carr and Madan[19] developed a higher-order difference method in his work. This work was targeted explicitly towards singularly perturbed parabolic partial differential equations, providing a valuable tool for solving Ordinary Differential Equations (ODEs). Odhiambo[20] and Black and Scholes[21], focused on a moving mesh refinement approach. They presented an optimally accurate and uniformly convergent computational method for a parabolic system. Boundary layers characterise this system and stem from reaction-diffusion problems with arbitrarily small diffusion terms. A research studies by Fouque et al.[22], Das et al.[23], Odhiambo et al.[24] and Da Fonseca et al.[25] showcased the parameter uniform optimal order numerical approximation applied to a particular class of singularly perturbed reaction-diffusion problems. These problems involve a small perturbation parameter. Finally, a paper by Odhiambo[26] discussed higher-order accurate approximations on equal distributed meshes. Their focus was on the mixed-type reaction-diffusion systems originating from boundary layers and exhibiting a multi-scale nature. These solutions will help in solving the proposed model.

The organization of this paper is as follows: Section 2 lays the foundation by introducing the concept of the Wishart process, elaborating on the definitions of the Wishart stochastic volatility models, and the generalized Wishart model. It further delves into the correlation structures and the infinitesimal generator specific to the multidimensional Wishart model. The focus is shifted toward the pricing mechanisms for the European call option in Section 3. The attainment of this objective is facilitated through the application of Fourier Transform methodologies and perturbation strategies. These tools formulate a pricing equation explicitly tailored for a European call option. Section 4 gives the numerical analysis of the results to verify the models. Lastly, Section 5 encapsulates the study's findings and highlights potential avenues for further research.

2. The Wishart process

Definition 1. Let W_t, $t \geq 0$ in a given martingale measure Q be a $n \times n$ matrix-indexed Wiener process. The Wishart matrix process is denoted by Σ, satisfies the following equation:

$$d\Sigma_t = (\beta QQ^T + M\Sigma_t + M\Sigma_t^T)\,dt + \sqrt{\Sigma_t}dW_tQ + Q^T dW_t^T\sqrt{\Sigma_t} \tag{1}$$

with $Q \in GL_n$(R) as the invertible matrix, M_n as the non-positive matrix, $\Sigma_0 \in \tilde{S}_n^+$ is non negative symmetric matrix with β real parameter and S_n is the price of security at time n.

The condition of $\beta > (n - 1)$ is measured to make sure existence and uniqueness of the $\Sigma_t \in \tilde{S}_n^+$ solution for equation (1) and eigen values of the solutions are all non-negative within $t \geq 0$ whenever $\Sigma_t \in \tilde{S}_n^+$.

2.1. Wishart implied volatility model in the securities exchange market

Benabid et al.[27] stated an arbitrage-free frictionless finance market and using the risk-neutral measure, the risky securities whose value dynamics from the quadratic variation following;

$$\begin{gathered}\frac{dS(t)}{S(t)} = rdt + Tr[\sqrt{\Sigma_t}dZ(t)], S_0 = s \\ d\Sigma_t = (\beta QQ^T + M\Sigma_t + M\Sigma_t^T)dt + \sqrt{\Sigma_t}dW_tQ + Q^TdW_t^T\sqrt{\Sigma_t}, \quad \Sigma_0 = \Sigma\end{gathered} \tag{2}$$

with r denoting risk less interest rate, trace is Tr, $Z \in M_n$ is Brownian matrix, and Σ_t being a set of symmetric $n \times n$ definite- positive matrices.

It is observed that security implied volatility has a trace of Σ_t matrix, that is multidimensional processes of $\Omega, M, Q \in M_n$, as well as $W_t \in M_n$ is Brownian matrix.

Das[28] and Shakti et al.[29] improved the Wishart process by offering a matrix analogue of the square root mean-reverting process. We consider M as a negative to make sure the mean-reverting property and positivity of the volatility with the parameter $\beta > n - 1$ for the existence and uniqueness of the solution.

2.2. Correlation analysis structure

W_t, Z_t are the two correlated Brownian matrices that gives a constant correlated matrix $R \in M_n$, in Das and Rana[30], that describes the structure of correlation for Z_t:

$$Zt := WtRT + BtpI - RR_T$$

where I denote identity matrix, T is transpose, and B_t is an independent matrix Wiener Process from W_t. The correlation structure is a Wiener Process *(checkout proof in Appendix)*.

2.3. Bi-variate Wishart stochastic process within Securities Exchange market

This section presents a newly proposed model, which is a multifactor framework incorporating two Wishart variance processes, also referred to as a generalized Wishart stochastic volatility model with a pair of dependence matrices. This model integrates double volatility components: the trace of the Wishart process, where its diagonal components are intended to steer the volatility dynamics. Within an arbitrage-free financial market and under the consideration of a risk-neutral measure, the dynamics of a risky security will be analyzed as follows:

$$
\begin{aligned}
\frac{dS(t)}{S(t)} &= rdt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t], \quad S_o = s \\
d\Sigma_t &= (\beta QQ^T + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t}dW_t Q + Q^T dW_t^T\sqrt{\Sigma_t}, \quad \Sigma_o = \Sigma \\
d\bar{\Sigma}_t &= (\bar{\beta}\bar{Q}\bar{Q}^T + \bar{M}\bar{\Sigma}_t + \bar{\Sigma}_t\bar{M}^T)dt + \sqrt{\bar{\Sigma}_t}d\bar{W}_t\bar{Q} + \bar{Q}^T d\bar{W}_t^T\sqrt{\bar{\Sigma}_t}, \quad \bar{\Sigma}_o = \bar{\Sigma}
\end{aligned}
\tag{3}
$$

where $\beta, \bar{\beta}$ defined as real parameters with $\beta, \bar{\beta} > n-1$, $Q, QM, \bar{M} \in M_n$, Q is defined as invertible matrix and $W_t, \bar{W}_t \in M_n$ called matrices Wiener Processes, also $Z_t, \bar{Z}_t \in M_n$.

Lemma 1: In the event where correlations are provided between two Brownian matrices—one belonging to the security price dynamics and the other associated with the Wishart processes, which possess Brownian matrices as expressed in equation (3)—then the following will be assumed:

$$
\rho_t = \frac{Tr(R^T Q\Sigma_t)}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^T Q\Sigma_t)}}
$$

$$
\bar{\rho}_t = \frac{Tr(\bar{R}^T\bar{Q}\bar{\Sigma}_t)}{\sqrt{Tr(\bar{\Sigma}_t)}\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}}
$$

Proof. The correlations of the process are derived as follows:

$$
\begin{aligned}
\frac{dS_t}{S_t} &= rdt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t] \\
&= rdt + \sqrt{Tr(\Sigma_t)}\frac{Tr(\sqrt{\Sigma_t}dZ_t)}{\sqrt{Tr(\Sigma_t)}} + \sqrt{Tr(\bar{\Sigma}_t)}\frac{Tr(\sqrt{\bar{\Sigma}_t}d\bar{Z}_t)}{\sqrt{Tr(\bar{\Sigma}_t)}} \\
&= rdt + \sqrt{Tr(\Sigma_t)}dX_t + \sqrt{Tr(\bar{\Sigma}_t)}d\bar{X}_t
\end{aligned}
$$

with X_t and $\bar{X}_t$ are standard Wiener Processes and trace respectively of the dynamics of Wishart volatility dynamics defined in equation (3). It becomes

$$
dTr(\Sigma_t) = ((\beta Tr(Q^T Q)) + 2Tr(M\Sigma_t))dt + 2Tr(QdW_t\Sigma_t)
$$

$$
dTr(\bar{\Sigma}_t) = ((\bar{\beta} Tr(\bar{Q}^T\bar{Q})) + 2Tr(\bar{M}\bar{\Sigma}_t))dt + 2Tr(\bar{Q}d\bar{W}_t\bar{\Sigma}_t)
$$

We rewrite the processes in the form

$$
dTr(\Sigma_t) = ((\beta Tr(Q^T Q)) + 2Tr(M\Sigma_t))dt + 2\sqrt{Tr(Q^T Q\Sigma_t)}\frac{Tr(QdW_t\Sigma_t)}{\sqrt{Tr(Q^T Q\Sigma_t)}}
$$

$$
dTr(\bar{\Sigma}_t) = ((\bar{\beta}Tr(\bar{Q}^T\bar{Q})) + 2Tr(\bar{M}\bar{\Sigma}_t))dt + 2\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}\frac{Tr(\bar{Q}d\bar{W}_t\bar{\Sigma}_t)}{\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}}
$$

with ξ_t and η_t being Wiener Processes, therefore

$$
dTr(\Sigma_t) = ((\beta Tr(Q^T Q)) + 2Tr(M\Sigma_t))dt + 2\sqrt{Tr(Q^T Q\Sigma_t)}d\xi_t
$$

$$
dTr(\bar{\Sigma}_t) = ((\bar{\beta}Tr(\bar{Q}^T\bar{Q})) + 2Tr(\bar{M}\bar{\Sigma}_t))dt + 2\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}d\eta_t
$$

We go ahead to find out the covariation of the generalized Wishart processes and security prices as;

$$
Cov_t(dX_t, d\xi_t) = Cov_t\left(\frac{Tr(\sqrt{\Sigma_t}dZ_t)}{\sqrt{Tr(\Sigma_t)}}, \frac{Tr(QdW_t\Sigma_t)}{\sqrt{Tr(Q^T Q\Sigma_t)}}\right)
$$

$$
\begin{aligned}
&= \mathbb{E}_t\left(\frac{Tr(\sqrt{\Sigma_t}dW_t R^T)}{\sqrt{Tr(\Sigma_t)}}\frac{Tr(QdW_t\Sigma_t)}{\sqrt{Tr(Q^T Q\Sigma_t)}}\right) = \frac{\sum_{ij} Cov_t(e_i^T\sqrt{\Sigma_t}dW_t R^T e_i, e_j^T QdW_t\sqrt{\Sigma_t}e_j)}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^T Q\Sigma_t)}} \\
&= \frac{\sum_{ij}\mathbb{E}_t(e_i^T\sqrt{\Sigma_t}dW_t R^T e_i e_j^T Q^T dW_t^T\sqrt{\Sigma_t}e_j)}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^T Q\Sigma_t)}} = \frac{\sum_{ij} e_i\sqrt{\Sigma_t}Tr(R^T e_i e_j^T Q^T)\sqrt{\Sigma_t}e_j dt}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^T Q\Sigma_t)}} \\
&= \frac{\sum_{ij} Tr(QRe_i e_j^T)e_i^T\Sigma_t e_j dt}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^T Q\Sigma_t)}} = \frac{\sum_{ij} e_j^T QRe_i e_i^T\Sigma_t e_j dt}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^T Q\Sigma_t)}}
\end{aligned}
$$

$$= \frac{\sum_{ij} e_j^T QR\Sigma_t e_j dt}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}}$$

$$Cov_t(dX_t, d\xi_t) = \frac{Tr(R^TQ\Sigma_t)}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}}$$

On the same note, for the second SDE (stochastic differential equation) of the generalized Wishart processes, its covariation determination that follow a similar procedure given as;

$$Cov_t(dX_t, d\eta_t) = \frac{Tr(\bar{R}^T\bar{Q}\bar{\Sigma}_t)}{\sqrt{Tr(\bar{\Sigma}_t)}\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}}$$

2.4. The generalized model correlation structure

W_t with Z_t and $\bar{W}_t$ with $\bar{Z}_t$ are correlated Brownian matrices respectively which results in constant correlation matrices $R, \bar{R} \in M_n$, which describes the double correlation structures, with Z_t and $\bar{Z}_t$ can be presented as;

$$Z_t := W_tR^T + B_t\sqrt{\mathbb{I} - RR^T}$$

$$\bar{Z}_t := \bar{W}_t\bar{R}^T + \bar{B}_t\sqrt{\mathbb{I} - \bar{R}\bar{R}^T}$$

With I as the identity matrix, T is the transposition, B_t and $\bar{B}_t$ being Brownian matrices independent of W_t and $\bar{W}_t$ in respectively of the generalized Wishart process.

2.5. The log-call price dynamics for the generalized Wishart model

The two matrices R and $\bar{R}$ respectively describes the correlations between the Brownian of the security and of the generalized Wishart processes. In addition, the value of M and $\bar{M}$ mean reversion matrices while $Q, \bar{Q}$ are implied volatility dynamics.

Lemma 2: The log-call price dynamics $Y_t = \log(S_t)$ under generalized Wishart and by application of the Ito's formula on Y_t, as proved by Das and Vigo-Aguiar[31] and Shakti et al[32], we will obtain

$$dY_t = (r - \frac{1}{2}Tr[\Sigma_t + \bar{\Sigma}_t])dt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t] \tag{4}$$

Proof. Let Y_t be $\log(S_t)$, the security dynamics defined as,

$$\frac{dS(t)}{S(t)} = rdt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t] \tag{5}$$

When we apply the Ito's formula on the value of Y_t

$$dY_t = d\log(S_t) = \frac{dS_t}{S_t} - \frac{1}{2}\frac{(dS_t)^2}{S_t^2} \tag{6}$$

Through substitution in the security process (5) within the derivative equation (6) of Y_t

$$dY(t) = rdt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t] - \frac{1}{2}Tr[\Sigma_t + \bar{\Sigma}_t]dt \tag{7}$$

$$dY_t = (r - \frac{1}{2}Tr[\Sigma_t + \bar{\Sigma}_t])dt + Tr[\sqrt{\Sigma_t}dZ_t + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t]$$

that will yield an equation given as

$$\begin{aligned} dY(t) = (r - \frac{1}{2}Tr[\Sigma_t + \bar{\Sigma}_t])dt + Tr[\sqrt{\Sigma_t}(dW_tR^T + dB_t\sqrt{\mathbb{I} - RR^T}) \\ + \sqrt{\bar{\Sigma}_t}(d\bar{W}_t\bar{R}^T + d\bar{B}_t\sqrt{\mathbb{I} - \bar{R}\bar{R}^T})], \quad Y_0 = y. \end{aligned} \tag{8}$$

3. Call option valuation problem

This section deals with the European call option valuation problem, with its payoff given as.

$$(S_T - K)^+$$

Addressing the pricing issue requires obtaining the infinitesimal generator of the Wishart processes. This approach is integral as it enables the application of the conditional characteristic function to the logarithmic return of the security. The Riccati ordinary differential equations are linearized to achieve a closed-form solution for this pricing problem. The issue at hand is addressed using Fast Fourier Transforms, a technique comprehensively delineated by Das et al.[33] .It's worth emphasizing that the Wishart processes retain a crucial feature of analytical tractability. This characteristic is due to their categorization under affine processes, an observation noted by Shreve[34].This aspect simplifies the analysis of these processes and enhances the computational efficiency of pricing and hedging instruments based on them. Combining these methods allows for a systematic and efficient approach to handling the pricing issue. It leads to deeper insights into how the multifactor Wishart stochastic volatility model operates and its implications for financial market dynamics. This methodology paves the way for further research, potential financial market modelling, and risk management improvements.

3.1. Generator of infinitesimal operator

The log-call price process and its volatility differential equations, having equivalent pair of two correlated Wiener Processes, Z_s^Y , Z_s^Σ and $\bar{Z}_s^Y$, $\bar{Z}_s^{\bar{\Sigma}}$, can be denoted in terms of determinants for simplicity when dealing of the complex securities of the dynamics given as follows;

$$\begin{aligned} dY_t &= (r - \frac{1}{2}Tr[\Sigma_t + \bar{\Sigma}_t])dt + Tr[\sqrt{\Sigma_t}dZ_t^Y + \sqrt{\bar{\Sigma}_t}d\bar{Z}_t^Y] \\ dTr(\Sigma_t) &= ((\beta Tr(Q^TQ)) + 2Tr(M\Sigma_t))dt + 2\sqrt{Tr(Q^TQ\Sigma_t)}(\rho_t dZ_t^Y + \sqrt{\mathbb{I} - \rho_t^2}dZ_t^\Sigma) \\ dTr(\bar{\Sigma}_t) &= ((\bar{\beta} Tr(\bar{Q}^T\bar{Q})) + 2Tr(\bar{M}\bar{\Sigma}_t))dt + 2\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}(\bar{\rho}_t d\bar{Z}_t^Y + \sqrt{\mathbb{I} - \bar{\rho}_t^2}d\bar{Z}_t^{\bar{\Sigma}}) \\ d < Z^Y, Z^\Sigma >_t &= \rho_t = \frac{Tr(R^TQ\Sigma_t)}{\sqrt{Tr(\Sigma_t)}\sqrt{Tr(Q^TQ\Sigma_t)}} \\ d < \bar{Z}^Y, \bar{Z}^{\bar{\Sigma}} >_t &= \bar{\rho}_t = \frac{Tr(\bar{R}^T\bar{Q}\bar{\Sigma}_t)}{\sqrt{Tr(\bar{\Sigma}_t)}\sqrt{Tr(\bar{Q}^T\bar{Q}\bar{\Sigma}_t)}} \end{aligned} \tag{9}$$

Proposition 1: Let the infinitesimal generator of the generalized Wishart stochastic volatility model for vector $(Y_t, \Sigma_t, \bar{\Sigma}_t)$ be defined as;

$$\begin{aligned} \mathcal{L}_{Y,\Sigma,\bar{\Sigma}} =& (r - \frac{Tr[\Sigma + \bar{\Sigma}]}{2})\frac{\partial}{\partial y} + \frac{Tr[\Sigma + \bar{\Sigma}]}{2}\frac{\partial^2}{\partial y^2} \\ &+ (\beta Tr(Q^TQ) + 2Tr(M\Sigma))\frac{\partial}{\partial \Sigma} + 2Tr(\Sigma\frac{\partial}{\partial \Sigma}Q^TQ\frac{\partial}{\partial \Sigma}) \\ &+ (\bar{\beta} Tr(\bar{Q}^T\bar{Q}) + 2Tr(\bar{M}\bar{\Sigma}))\frac{\partial}{\partial \bar{\Sigma}} + 2Tr(\bar{\Sigma}\frac{\partial}{\partial \bar{\Sigma}}\bar{Q}^T\bar{Q}\frac{\partial}{\partial \bar{\Sigma}}) \\ &+ 2Tr(\Sigma RQ\frac{\partial}{\partial \Sigma})\frac{\partial}{\partial y} + 2Tr(\bar{\Sigma}\bar{R}\bar{Q}\frac{\partial}{\partial \bar{\Sigma}})\frac{\partial}{\partial y} \end{aligned} \tag{10}$$

Proof. The generator of infinitesimal having a non-trivial term will arise from covariation $d < \Sigma_\theta^{ij}, Y >$ corresponding up to the terms' coefficients defined as;

$$\frac{\partial^2}{\partial x_{\theta;ij}} = \frac{\partial^2}{\partial x_{\theta;ij}}(\frac{\partial}{\partial y}), \quad i,j = 1,...,n, \quad \theta = 1,2$$

Let $V_{\theta;t} := {}^{\mathrm{p}}\Sigma_{\theta;t}$ be denoted as the square root matrix with

$$\Sigma_{\theta;t}^{ij} = \sum_{t=1}^{n} V_{\theta;t}^{il}V_{\theta;t}^{lj} = \sum_{t=1}^{n} V_{\theta;t}^{il}V_{\theta;t}^{jl}$$

From the value of $V_{\theta;t}$ that is symmetric, we determine the covariation terms that matches with $\frac{\partial^2}{\partial x_{\theta;ij}\partial y}$ coefficients

$$< d\Sigma_\theta^{ij}, Y > = \mathbb{E}_t[(\sum_{i,k=1}^{n} V_{\theta;t}^{il} dW_{lk}^\theta Q_{kj}^\theta + \sum_{l,k=1}^{n} V_{\theta;t}^{jl} dW_{lk}^\theta Q_{ki}^\theta)(\sum_{l,k,h=1}^{n} V_{\theta;t}^{lk} dW_{kh}^\theta R_{lh}^\theta)]$$
$$= \sum_{l,k,h=1}^{n} (V_{\theta;t}^{il} Q_{kj}^\theta + V_{\theta;t}^{jl} Q_{ki}^\theta) V_{\theta;t}^{hl} R_{hk}^\theta dt$$
$$= \sum_{k,h=1}^{n} [(\sum_{l=1}^{n} V_{\theta;t}^{il} V_{\theta;t}^{hl}) Q_{kj}^\theta + (\sum_{l=1}^{n} V_{\theta;t}^{jl} V_{\theta;t}^{hl}) Q_{ki}^\theta] R_{hk}^\theta dt$$
$$= \sum_{k,h=1}^{n} (\Sigma_{\theta;t}^{ih} Q_{kj}^\theta + \Sigma_{\theta;t}^{jh} Q_{ki}^\theta) R_{hk}^\theta dt$$

It leads to corresponding term coefficients as it is given by;

$$2Tr(\Sigma_\theta R^\theta Q^\theta D_\theta)\frac{\partial}{\partial y} = 2\sum_{i,j,k,h=1}^{n} D_\theta^{ij} \Sigma_\theta^{jh} R_{hk}^\theta Q_{ki}^\theta \frac{\partial}{\partial y}$$

The notation occurs when

$$\theta = 1, \Sigma_1 = \Sigma, R^1 = R, Q^1 = Q, D_1 = \frac{\partial}{\partial x_{ij}}$$

whereas for $\theta = 2$, $\Sigma_2 = \bar{\Sigma}, R^2 = \bar{R}, Q^2 = \bar{Q}, D_2 = \frac{\partial}{\partial \bar{x}_{ij}}$, as the value of D is symmetric.

3.2. The security returns Laplace transformations

To address the pricing problem of European options for dynamic process (10), we use the transforms introduced by Das et al.[35]. Because the Wishart process is exponentially affine, as noted in studies by Das et al.[36] introduced the conditional expectation of security returns can be expressed as an affine exponential of Y and the components of the Wishart process. As such, we put forward deterministic functions λ1(t), λ2(t) in the Mn set and δ(t), ε(t) in the R set, serving as parameters for the transformation. This approach allows for a thorough and accurate representation of the factors influencing European options pricing. The equation is given by;

$$\psi\gamma,t(\tau) = \mathrm{E}(e\gamma Yt+\tau) = exp\{Tr\lambda_1(\tau)\Sigma_t + Tr\lambda_2(\tau)\bar{\Sigma}_t + \delta(\tau)Y_t + \varepsilon(\tau)\} \tag{11}$$

with $\gamma \in$ R.

From the Feynman-Kac argument given in equation (12), assist in obtaining matrix Riccati equations given as;

$$\begin{cases} \frac{\partial \psi_{\gamma,t}}{\partial \tau} & = \mathcal{L}_{Y,\Sigma,\bar{\Sigma}} \psi_{\gamma,t} \\ \psi_{\gamma,t}(0) & = exp\{\gamma Y_t\} \end{cases} \tag{12}$$

Proposition 2: Let solution of the stochastic differential equation of the security returns under Laplace Transforms be given by

$$\psi_{\gamma,t}(\tau) = exp\{Tr[\lambda_1(\tau)\Sigma_t + \lambda_2(\tau)\bar{\Sigma}_t] + \delta(\tau)Y_t + \varepsilon(\tau)\}$$

where λ_1, λ_2, ε will be solutions to the differential equations below

$$\lambda_1'(\tau) = M\lambda_1(\tau) + (M^T + 2\gamma RQ)\lambda_1(\tau) + 2\lambda_1(\tau)Q^TQ\lambda_1(\tau) + \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n \tag{13}$$

$$\begin{gathered} \lambda_2'(\tau) = \bar{M}\lambda_2(\tau) + (\bar{M}^T + 2\gamma\bar{R}\bar{Q})\lambda_2(\tau) + 2\lambda_2(\tau)\bar{Q}^T\bar{Q}\lambda_2(\tau) + \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n \\ \varepsilon'(\tau) = r\gamma + \beta Tr[(Q^TQ)\lambda_1(\tau)] + \bar{\beta}Tr[(\bar{Q}^T\bar{Q})\lambda_2(\tau)] \end{gathered} \tag{14}$$

with boundary conditions; $\lambda_1(0) = 0$, $\lambda_2(0) = 0$, $\varepsilon(0) = 0, \delta(\tau) = \gamma = C_0$. All solutions of λ_1, λ_2, ε, are derived as follows

$$\lambda_1(\tau) = H_1(\tau)^{-1}H_2(\tau)$$
$$\lambda_2(\tau) = I_1^{-1}(\tau)I_2(\tau)$$
$$\varepsilon(\tau) = -\frac{\beta}{2}Tr[\log H_1(\tau) + (M^T + 2\gamma RQ)\tau] - \frac{\bar{\beta}}{2}Tr[\log I_1(\tau) + (\bar{M}^T + 2\gamma\bar{R}\bar{Q})\tau] + r\gamma\tau \tag{15}$$

with

$$\begin{aligned}(H_2(\tau) \quad H_1(\tau)) &= (H_2(0) \quad H_1(0))exp\tau\begin{pmatrix} M & -2Q^TQ \\ \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n & -(M^T + 2\gamma RQ)\end{pmatrix} \\ (I_2(\tau) \quad I_1(\tau)) &= \begin{pmatrix} I_2(0) & I_1(0)\end{pmatrix} exp\tau\begin{pmatrix} \bar{M} & -2\bar{Q}^T\bar{Q} \\ \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n & -(\bar{M}^T + 2\gamma\bar{R}\bar{Q})\end{pmatrix}\end{aligned} \tag{16}$$

Proof. From the Laplace Equation (12), we take up the computation of the problem through consideration of proposition (1) for the generalized Wishart stochastic implied volatility model, thus offers the proof.

$$\begin{aligned}\frac{\partial\psi_{\gamma,t}}{\partial\tau} &= (r - \frac{Tr[\Sigma+\bar{\Sigma}]}{2})\frac{\partial\psi_{\gamma,t}}{\partial y} + \frac{Tr[\Sigma+\bar{\Sigma}]}{2}\frac{\partial^2\psi_{\gamma,t}}{\partial y^2} \\ &+ (\beta Tr(Q^TQ) + 2Tr(M\Sigma))\frac{\partial\psi_{\gamma,t}}{\partial\Sigma} + 2Tr(\Sigma\frac{\partial}{\partial\Sigma}Q^TQ\frac{\partial}{\partial\Sigma})\psi_{\gamma,t} \\ &+ (\bar{\beta}Tr(\bar{Q}^T\bar{Q}) + 2Tr(\bar{M}\bar{\Sigma}))\frac{\partial\psi_{\gamma,t}}{\partial\bar{\Sigma}} + 2Tr(\bar{\Sigma}\frac{\partial}{\partial\bar{\Sigma}}\bar{Q}^T\bar{Q}\frac{\partial}{\partial\bar{\Sigma}})\psi_{\gamma,t} \\ &+ 2Tr(\Sigma RQ\frac{\partial}{\partial\Sigma})\frac{\partial\psi_{\gamma,t}}{\partial y} + 2Tr(\bar{\Sigma}\bar{R}\bar{Q}\frac{\partial}{\partial\bar{\Sigma}})\frac{\partial\psi_{\gamma,t}}{\partial y}\end{aligned} \tag{17}$$

Taking into consideration of Boundary conditions in (11) and equation (12) $\lambda_1(0) = 0$, $\lambda_2(0) = 0$, $\delta(0) = \gamma$, $\varepsilon(0) = 0$

$$\begin{aligned}\frac{\partial\psi_{\gamma,t}(\tau)}{\partial\tau} &= Tr\{(\frac{d\lambda_1(\tau)}{d\tau}\Sigma_t + \frac{d\lambda_2(\tau)}{\partial\tau}\bar{\Sigma}_t) + \frac{d}{d\tau}\delta(t)Y_t + \frac{d}{d\tau}\varepsilon(\tau)\}\psi_{\gamma,t}(\tau) \\ &= (r - \frac{Tr[\Sigma+\bar{\Sigma}]}{2})\delta(\tau)\psi_{\gamma,t} + \frac{Tr(\Sigma+\bar{\Sigma})}{2}\delta^2(\tau)\psi_{\gamma,t} \\ &+[(\beta Tr(Q^TQ) + 2Tr(M\Sigma))\lambda_1(\tau) + (\bar{\beta}Tr(\bar{Q}^T\bar{Q}) + 2Tr(\bar{M}\bar{\Sigma}))\lambda_2(\tau)]\psi_{\gamma,t} \\ &+2Tr[\Sigma\lambda_1(\tau)Q^TQ\lambda_1(\tau)]\psi_{\gamma,t}(\tau) + 2Tr[\bar{\Sigma}\lambda_1(\tau)\bar{Q}^T\bar{Q}\lambda_1(\tau)]\psi_{\gamma,t} \\ &+2Tr[\Sigma RQ\delta(\tau)\lambda_1(\tau)]\psi_{\gamma,t} + 2Tr[\bar{\Sigma}\bar{R}\bar{Q}\delta(\tau)\lambda_1(\tau)]\psi_{\gamma,t}.\end{aligned} \tag{18}$$

$$\begin{aligned}&-[Tr(\acute{\lambda}_1(\tau)\Sigma_t + \acute{\lambda}_2(\tau)\bar{\Sigma}_t) + \acute{\delta}(\tau)Y_t + \acute{\varepsilon}(\tau)] + (r - \frac{Tr[\Sigma+\bar{\Sigma}]}{2})\delta(\tau) \\ &+\frac{Tr(\Sigma+\bar{\Sigma})}{2}\delta^2(\tau) + Tr[(\beta Q^TQ + M\Sigma + \Sigma M^T)\lambda_1(\tau) + (\bar{\beta}\bar{Q}^T\bar{Q} + \bar{M}\bar{\Sigma} + \bar{\Sigma}\bar{M}^T)\lambda_2(\tau) \\ &+2\Sigma\lambda_1(\tau)Q^TQ\lambda_1(\tau) + 2\bar{\Sigma}\lambda_2(\tau)\bar{Q}^T\bar{Q}\lambda_2(\tau) + 2\Sigma RQ\delta(\tau)\lambda_1(\tau) + 2\bar{\Sigma}\bar{R}\bar{Q}\delta(\tau)\lambda_2(\tau)] = 0\end{aligned} \tag{19}$$

$$\begin{aligned}&-Tr(\frac{d}{d\tau}\lambda_1(\tau)\Sigma) - \frac{1}{2}Tr[\Sigma]\delta(\tau) + \frac{1}{2}Tr[\Sigma]\delta^2(\tau) \\ &+Tr[(M\Sigma + \Sigma M^T)\lambda_1(\tau) + 2\Sigma\lambda_1(\tau)Q^TQ\lambda_1(\tau) + 2\Sigma RQ\delta(\tau)\lambda_1(\tau)] = 0\end{aligned} \tag{20}$$

$$\begin{aligned}&-Tr(\frac{d}{d\tau}\lambda_2(\tau)\bar{\Sigma}) - \frac{1}{2}Tr[\bar{\Sigma}]\delta(\tau) + \frac{1}{2}Tr[\bar{\Sigma}]\delta^2(\tau) \\ &+Tr[(\bar{M}\bar{\Sigma} + \bar{\Sigma}\bar{M}^T)\lambda_2(\tau) + 2\bar{\Sigma}\lambda_2(\tau)\bar{Q}^T\bar{Q}\lambda_2(\tau) + 2\bar{\Sigma}\bar{R}\bar{Q}\delta(\tau)\lambda_2(\tau)] = 0\end{aligned} \tag{21}$$

$$-\delta'(\tau)Y_t - \varepsilon'(\tau) + \mu\delta(\tau) + Tr[\beta Q^TQ]\lambda_1(\tau) + Tr[\bar{\beta}\bar{Q}^T\bar{Q}]\lambda_2(\tau) = 0 \tag{22}$$

Now we proceed to make identification of the coefficients for the respective equations and by deriving the matrix Riccati ODE (ordinary differential equations);

$$\begin{cases} \frac{d}{d\tau}\lambda_1(\tau) & = M\lambda_1(\tau) + (M^T + 2\gamma RQ)\lambda_1(\tau) + 2\lambda_1(\tau)Q^TQ\lambda_1(\tau) + \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n \\ \lambda_1(0) & = 0 \end{cases} \tag{23}$$

$$\begin{cases} \frac{d}{d\tau}\lambda_2(\tau) & = \bar{M}\lambda_2(\tau) + (\bar{M}^T + 2\gamma \bar{R}\bar{Q})\lambda_2(\tau) + 2\lambda_2(\tau)\bar{Q}^T\bar{Q}\lambda_2(\tau) + \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n \\ \lambda_2(0) & = 0 \end{cases} \tag{24}$$

For the constant value of ε, the matrix Riccati ordinary differential is derived as follows

$$-\delta'(\tau)Y_t - \varepsilon'(\tau) + r\delta(\tau) + Tr(\beta Q^T Q)\lambda_1(\tau) + Tr(\bar{\beta}\bar{Q}^T\bar{Q})\lambda_2(\tau) = 0$$

$$\delta(\tau) = C_0 = \gamma, \quad since \quad \delta'(\tau) = 0 \quad so \Rightarrow \delta(\tau) = C_0 = \gamma$$

$$\begin{cases} \frac{d\varepsilon(\tau)}{d\tau} = r\gamma + \beta Tr[(Q^TQ)\lambda_1(\tau)] + \bar{\beta}Tr[(\bar{Q}^T\bar{Q})\lambda_2(\tau)] \\ \varepsilon(0) = 0 \end{cases} \tag{25}$$

Lastly, $\varepsilon(\tau)$ is derived through integration directly

$$\varepsilon(\tau) = \int_0^\tau r\gamma ds + \int_0^\tau Tr[\beta Q^TQ\lambda_1(s) + \bar{\beta}\bar{Q}^T\bar{Q}\lambda_2(s)]ds$$

We note that $\lambda_1, \lambda_2 \in M_n(\mathrm{R})$ as well as $\delta(\tau), \varepsilon(\tau) \in \mathrm{R}$.

Following the methodologies proposed by Das et al.[37], we proceed to linearize the matrix Riccati equations previously mentioned. This step enables us to derive a closed-form solution. This solution results from the systematic application of Equations (23) and (24), providing a comprehensive and practical approach to understand the underpinnings of the model.

Let $\lambda_1(\tau) = H_1(\tau)^{-1}H_2(\tau)$ with $H_1(\tau) \in GL_n(\mathrm{R})$ and $H_2(\tau) \in M_n(\mathrm{R})$ and thus

$$\begin{aligned} \frac{d}{d\tau}[H_1(\tau)\lambda_1(\tau)] &= \frac{dH_1(\tau)}{d\tau}\lambda_1(\tau) + H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau} \\ H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau} &= \frac{d}{d\tau}[H_1(\tau)\lambda_1(\tau)] - \frac{dH_1(\tau)}{d\tau}\lambda_1(\tau) \end{aligned} \tag{26}$$

$$\begin{aligned} H_1(\tau)\lambda_1(\tau)M + H_1(\tau)(M^T + 2\gamma RQ)\lambda_1(\tau) + 2H_1(\tau)\lambda_1(\tau)Q^TQ\lambda_1(\tau) \\ + H_1(\tau)\frac{\gamma(\gamma-1)}{2} = H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau} \end{aligned} \tag{27}$$

$$\begin{aligned} H_2(\tau)M + H_1(\tau)(M^T + 2\gamma RQ)\lambda_1(\tau) + 2H_2(\tau)Q^TQ\lambda_2(\tau) \\ + H_1(\tau)\frac{\gamma(\gamma-1)}{2} = H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau} \end{aligned} \tag{28}$$

Given that $H_2(\tau) = H_1(\tau)\lambda_1(\tau)$ now we will have

$$\frac{dH_2(\tau)}{d\tau} = \frac{dH_1(\tau)}{d\tau}\lambda_1(\tau) + H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau}$$

The solution implies that

$$H_1(\tau)\frac{d\lambda_1(\tau)}{d\tau} = \frac{dH_2(\tau)}{d\tau} - \frac{dH_1(\tau)}{d\tau}\lambda_1(\tau) \tag{29}$$

Therefore, we obtain an expression

$$\begin{aligned} \frac{dH_2(\tau)}{d\tau} - \frac{dH_1(\tau)}{d\tau}\lambda_1(\tau) &= H_2(\tau)M + H_1(\tau)(M^T + 2\gamma RQ)\lambda_1(\tau) + 2H_2(\tau)Q^TQ\lambda_1(\tau) \\ &\quad + H_1(\tau)\frac{\gamma(\gamma-1)}{2} \\ &= H_1(\tau)\frac{\gamma(\gamma-1)}{2} + H_2(\tau)M + [H_1(\tau)(M^T + 2\gamma RQ) + 2H_2(\tau)Q^TQ]\lambda_1(\tau) \end{aligned} \tag{30}$$

$$\frac{dH_2(\tau)}{d\tau} = H_1(\tau)\frac{\gamma(\gamma-1)}{2} + H_2(\tau)M \tag{31}$$

$$\frac{dH_1(\tau)}{d\tau} = -H_1(\tau)(M^T + 2\gamma RQ) - 2H_2(\tau)Q^TQ \tag{32}$$

$$\frac{d}{d\tau}(H_2(\tau) \quad H_1(\tau)) = \begin{pmatrix} H_2(\tau) & H_1(\tau) \end{pmatrix} \begin{pmatrix} M & -2Q^TQ \\ \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n & -(M^T + 2\gamma RQ) \end{pmatrix} \tag{33}$$

The differential equation solution as in Equation (33) above yields;

$$\begin{pmatrix} H_2(\tau) & H_1(\tau) \end{pmatrix} = \begin{pmatrix} H_2(0) & H_1(0) \end{pmatrix} e^{Q\tau} \tag{34}$$

under the conditions $H_1(0) = K\mathrm{I}n$ and $H_1^{-1}(0) = K^{-1}\mathbb{I}_n$ and for $K = 1, \quad H_1^{-1}(0) = \mathbb{I}_n$

we have

$$(H_2(\tau)\ H_1(\tau)) = (\lambda_1(0)\ \mathrm{I}n)e^{\tau Q}$$

$$e^{\tau Q} = \begin{pmatrix} \lambda_1^{11}(\tau) & \lambda_1^{12}(\tau) \\ \lambda_1^{21}(\tau) & \lambda_1^{22}(\tau) \end{pmatrix}$$

$\lambda 111\ (\tau)$ $\qquad\qquad$ $\lambda 121\ (\tau)!$

$$(H2(\tau) \quad H1(\tau)) = (\lambda 1(0)\ \mathrm{I}n) \qquad 21(\tau) \quad \lambda 22_1(\tau)\ \lambda_1 \tag{35}$$

$$= [\lambda_1(0)\lambda^{11}{}_1(\tau) + \lambda^{21}{}_1(\tau) \quad \lambda_1(0)\lambda^{12}{}_1(\tau) + \lambda^{22}{}_1(\tau)]$$

and since $\lambda_1(0) = 0$, then

$$(H_2(\tau) \quad H_1(\tau)) = (\lambda_1^{21}(\tau) \quad \lambda_1^{22}(\tau))$$

accordingly,

$$\lambda_1(\tau) = \lambda_1^{22}(\tau)^{-1}\lambda_1^{21}(\tau) \tag{36}$$

It is first Riccati Equation (23) closed-form solution.

Now we relook at the second Riccati Equation (24) solution.

Definition 2: Let equation (37) be given by:

$$\lambda_2(\tau) = I_1^{-1}(\tau)I_2(\tau) \tag{37}$$

then

$$\frac{dI_2(\tau)}{d\tau} = I_1(\tau)\frac{\gamma(\gamma-1)}{2} + I_2(\tau)\bar{M}$$

$$\frac{dI_1(\tau)}{d\tau} = -I_1(\tau)(\bar{M}^T + 2\gamma\bar{R}\bar{Q}) - 2I_2(\tau)\bar{Q}^T\bar{Q} \tag{38}$$

$$\frac{d}{d\tau}(I_2(\tau) \quad I_1(\tau)) = \begin{pmatrix} I_2(\tau) & I_1(\tau) \end{pmatrix} \begin{pmatrix} \bar{M} & -2\bar{Q}^T\bar{Q} \\ \frac{\gamma(\gamma-1)}{2}\mathbb{I}_n & -(\bar{M}^T + 2\gamma\bar{R}\bar{Q}) \end{pmatrix}$$

We have,

$$(I_2(\tau) \qquad I_1(\tau)) = (I_2(0) \quad I_1(0))e^{Q\tau^-} \tag{39}$$

From equation (39), we repeat a similar procedure as given above having similar conditions as

$$I_2(0) = I_1(0)\lambda_2(0) = \lambda_2(0)$$

$$I_1(0) = \bar{K}\mathrm{I} = \bar{K} = \mathrm{I}$$

thus

$$(I_2(\tau) \qquad I_1(\tau)) = (\lambda_2(0) \qquad \mathrm{I}n) \begin{pmatrix} \lambda_2^{11}(\tau) & \lambda 122\ (\tau)!\ \lambda 222\ (\tau) \\ \lambda_2^{21}(\tau) & \end{pmatrix} \tag{40}$$

$$(I_2(\tau)I_1(\tau)) = (\lambda_2(0)\lambda^{11}{}_2(\tau) + \lambda^{21}{}_2(\tau) \rightarrow \lambda_2(0)\lambda_2^{12}(\tau) + \lambda_2^{22}(\tau)) \tag{41}$$

and since $\lambda_2(0) = 0$, then

$$\begin{pmatrix} I_2(\tau) & I_1(\tau) \end{pmatrix} = \begin{pmatrix} \lambda_2^{21}(\tau) & \lambda_2^{22}(\tau) \end{pmatrix}$$

thus

$$\lambda_2(\tau) = \lambda_2^{22}(\tau)^{-1}\lambda_2^{21}(\tau) \tag{42}$$

Let the computation of the last Riccati equation (25) for a constant ε ,

$$\begin{cases} \frac{d\varepsilon(\tau)}{d\tau} = r\gamma + \beta Tr[(Q^TQ)\lambda_1(\tau)] + \bar{\beta}Tr[(\bar{Q}^T\bar{Q})\lambda_2(\tau)] \\ \varepsilon(0) = 0 \end{cases} \tag{43}$$

From two Riccati equations (23) and (24) respectively, we get

$$\begin{cases} \frac{dH_2(\tau)}{d\tau} = H_1(\tau)\frac{\gamma(\gamma-1)}{2} + H_2(\tau)M \\ \frac{dH_1(\tau)}{d\tau} = -H_1(\tau)(M^T + 2\gamma RQ) - 2H_2(\tau)Q^TQ \end{cases} \tag{44}$$

and

$$\begin{cases} \frac{dI_2(\tau)}{d\tau} = I_1(\tau)\frac{\gamma(\gamma-1)}{2} + I_2(\tau)\bar{M} \\ \frac{dI_1(\tau)}{d\tau} = -I_1(\tau)(\bar{M}^T + 2\gamma\bar{R}\bar{Q}) - 2I_2(\tau)\bar{Q}^T\bar{Q} \end{cases} \tag{45}$$

$$\frac{d\varepsilon(\tau)}{d\tau} = r\gamma + \beta Tr[(Q^TQ)H_1^{-1}(\tau)H_2(\tau)] + \bar{\beta}Tr[(\bar{Q}^T\bar{Q})I_1^{-1}(\tau)I_2(\tau)] \tag{46}$$

Deriving from the equation (44) we have;

$$H_2(\tau) = -\frac{1}{2}[\frac{dH_1(\tau)}{d\tau} + H_1(\tau)(M^T + 2\gamma RQ)](Q^TQ)^{-1} \tag{47}$$

And from the equation (45)

$$I_2(\tau) = -\frac{1}{2}[\frac{dI_1(\tau)}{d\tau} + I_1(\tau)(\bar{M}^T + 2\gamma\bar{R}\bar{Q})](\bar{Q}^T\bar{Q})^{-1} \tag{48}$$

then substituting both equations (47) and (48) within the equation (46), results to

$$\begin{cases} \frac{d\varepsilon(\tau)}{d\tau} = -\frac{\beta}{2}Tr[H_1^{-1}(\tau)\frac{dH_1(\tau)}{d\tau} + (M^T + 2\gamma RQ)] - \frac{\bar{\beta}}{2}Tr[I_1^{-1}(\tau)\frac{dI_1(\tau)}{d\tau} + (\bar{M}^T + 2\gamma\bar{R}\bar{Q})] + r\gamma \\ \varepsilon(0) = 0 \end{cases} \tag{49}$$

$$d\varepsilon(\tau) = -\frac{\beta}{2}Tr[\frac{dH_1(\tau)}{H_1(\tau)} + (M^T + 2\gamma RQ)d\tau] - \frac{\bar{\beta}}{2}Tr[\frac{dI_1(\tau)}{I_1(\tau)} + (\bar{M}^T + 2\gamma\bar{R}\bar{Q})d\tau] + r\gamma d\tau \tag{50}$$

Through integration of the equation (50), we get

$$\varepsilon(\tau) = -\frac{\beta}{2}Tr[\log H_1(\tau) + (M^T + 2\gamma RQ)\tau] - \frac{\bar{\beta}}{2}Tr[\log I_1(\tau) + (\bar{M}^T + 2\gamma\bar{R}\bar{Q})\tau] + r\gamma\tau \tag{51}$$

3.3. The characteristic function and Fast Fourier transformation method

In this section, we utilize the Fast Fourier Transform (FFT) method as specified in Odhiambo et al.[38] work, aiming to calculate the price of a European call option. In this context, 'α' is a value exceeding zero, 't' signifies time, and the strike price is represented as 'k', which corresponds to the logarithm of 'K'. Furthermore, 'T' signifies the time to maturity. The intention is to provide a comprehensive calculation of the option's value under the aforementioned conditions as

$$C_t(T,K) = e^{-r(T-t)}\mathrm{E}[(X_T - K)^+ \mid \mathrm{F}_t] \tag{52}$$

$$C_t(T,K) = C_t(T, k) = e^{-r(T-t)}\mathrm{E}[(exp(Y_t) - exp(k))^+ | \mathrm{F}_t] \tag{53}$$

We turn our attention to the adjusted Price formulation found in DasP et al.[38], where α = 1.1 serves as a sound empirical figure applicable to the Heston model. By utilizing this modified price, we achieve a function that is square integrable. This function subsequently aids in the application of the inverse Fourier transform, thereby facilitating our calculation processes and enabling the successful execution of the pricing model, C_t^{α} given by;

$$C_t^{\alpha}(T,k) = exp(\alpha k)C_t(T,K) \tag{54}$$

We incorporate the Fourier transformation of the modified price while simultaneously employing the Fubini integration theorem:

$$\begin{aligned}\psi_t^{\alpha}(T,\theta) &= \int_{-\infty}^{+\infty} C_t^{\alpha}(T,k)e^{i\theta k}dk \\ &= \int_{-\infty}^{+\infty} e^{\alpha k}C_t(T,K)e^{i\theta k}dk\end{aligned} \tag{55}$$

$$= \int_{-\infty}^{+\infty} e^{\alpha k}e^{-r(T-t)}\mathbb{E}[(exp(Y_t) - exp(k))^+|\mathcal{F}_t]e^{i\theta k}dk$$

$$= e^{-r(T-t)}\int_{-\infty}^{+\infty} exp[(\alpha + i\theta)k]\mathbb{E}[(exp(Y_t) - exp(k))^+|\mathcal{F}_t]dk \tag{56}$$

$$\psi_t^{\alpha}(T,\theta) = \frac{e^{-r(T-t)}\phi_t(T,\theta-(1+\alpha)i)}{(\alpha+i\theta)(\alpha+i\theta+1)}$$

We determine the price of a call option through the inversion of the Fourier transform, provided that the ψtα(T,θ) function encompasses both odd imaginary and even real parts. This process can be illustrated by revisiting equation (56).

Getting

$$\int_{-\infty}^{+\infty} C_t^{\alpha}(T,k)e^{i\theta k}dk = \frac{e^{-r(T-t)}\phi_t(T,\theta-(1+\alpha)i)}{(\alpha+i\theta)(\alpha+i\theta+1)}$$

$$\begin{aligned}C_t^{\alpha}(T,k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_t^{\alpha}(T,\theta)e^{-i\theta k}d\theta \\ C_t(T,K) &= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} \psi_t^{\alpha}(T,\theta)e^{-i\theta k}d\theta\end{aligned} \tag{57}$$

$$C_t(T,K) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-r(T-t)}\tilde{\phi}_t(T,\theta-(1+\alpha)i)}{(\alpha+i\theta)(\alpha+i\theta+1)}e^{-i\theta k}d\theta$$

It is a Fourier transform

$$C_t(T,K) = \frac{e^{-\alpha k}}{2\pi}Re\left(\int_{-\infty}^{\infty} \frac{e^{-r(T-t)}\tilde{\phi}_t(T,\theta-(1+\alpha)i)}{(\alpha+i\theta)(\alpha+i\theta+1)}e^{-i\theta k}d\theta\right. \tag{58}$$

Corollary 1: Let D be defined as symmetric matrix, it is sufficient to determine the conditional characteristic function of the generalized Wishart Σ_t and $\bar{\Sigma}_t$ denoted by, (see proof in Appendix)

$$\begin{aligned}\tilde{\phi}_{\Sigma_t,\bar{\Sigma}_t}^{D_1D_2} &= \mathbb{E}\left[exp\{iTr[D_1\Sigma_t + D_2\bar{\Sigma}_t]\}\right] \\ &= exp\{Tr[A_1(\tau)\Sigma_t + A_2(\tau)\bar{\Sigma}_t] + C(\tau)\}\end{aligned} \tag{59}$$

Where values of $A_1(\tau)$, $A_2(\tau) \in M_n$ *and* $C \in$ C are used when verifying the following dynamics

$$\begin{cases} \frac{dA_1(\tau)}{d\tau} = A_1(\tau)M + M^T A_1(\tau) + 2A_1(\tau)Q^T Q A_1(\tau) \\ A_1(0) = iD_1 \end{cases}$$

$$\begin{cases} \frac{dA_2(\tau)}{d\tau} = A_2(\tau)\bar{M} + \bar{M}^T A_2(\tau) + 2A_2(\tau)\bar{Q}^T \bar{Q} A_2(\tau) \\ A_2(0) = iD_2 \end{cases} \tag{60}$$

$$C(\tau) = \int_0^\tau Tr[\beta Q^T Q A_1(\tau)du + \bar{\beta}\bar{Q}^T \bar{Q} A_2(\tau)du]$$

Proposition 3: Let the call option price under generalized Wishart is given as

$$C_t(T,K) = \frac{e^{-\alpha k}}{2\pi} Re\left(\int_{-\infty}^{\infty} e^{-r(T-t)}\tilde{\phi}_{\Sigma,\bar{\Sigma}}(t)\phi(\theta)e^{-i\theta k}d\theta\right) \tag{61}$$

whereas

$$\tilde{\phi}_{\Sigma,\bar{\Sigma}}(t) = e^{\{Tr[A_1(t)\Sigma_0 + A_2(t)\bar{\Sigma}_0] + C(t) + \varepsilon(T-t)\}}$$

$$\phi(\theta) = \frac{1}{(\alpha + i\theta)(\alpha + i\theta + 1)}$$

Proof. let $\tilde{\varphi}(t,T)$ be the log-call price, Y_t as the characteristic function. We will have,

$$Y_{t,T} = \ln\left(\frac{X_T}{X_t}\right) = \ln(X_T) - \ln(X_t) = Y_T - Y_t$$

and

$$\tilde{\varphi}_{\gamma,0}(t,T) = \mathrm{E}[exp\{i\gamma Y_{t,T}\}] = \mathrm{E}[\mathrm{E}exp\{i\gamma(Y_T - Y_t)\}]$$

$$= \mathbb{E}[(exp(-i\gamma Y_t))\mathbb{E}_t exp\{i\gamma Y_T\}]$$

$$= \mathbb{E}[exp(-i\gamma Y_t)exp\{Tr\left[\lambda_1(T-t)\Sigma_t + \lambda_2(T-t)\bar{\Sigma}_t\right] + i\gamma Y_t + \varepsilon(T-t)\}]$$

$$= exp\{\varepsilon(T-t)\}\mathbb{E}\left[exp\{Tr[\lambda_1(T-t)\Sigma_t + \lambda_2(T-t)\bar{\Sigma}_t]\}\right]$$

$$= exp\{\varepsilon(T-t)\}exp\{Tr[A_1(\tau)\Sigma_0 + A_2(\tau)\bar{\Sigma}_0] + C(\tau)\}$$

$$= exp\{\varepsilon(T-t)\}exp\{Tr[A_1(t)\Sigma_0 + A_2(t)\bar{\Sigma}_0] + C(t)\}$$

$$\tilde{\varphi}_{\gamma,0}(t,T) = exp\{Tr[A_1(t)\Sigma_0 + A_2(t)\bar{\Sigma}_0] + C(t) + \varepsilon(T-t)\}$$

$A_j(t)$ is got from the equation (61) with $\tau = t$ if $A_j(0) = \lambda_j(T-t)$, $j = 1,2$.

Therefore

$$C_t(T,K) = \frac{e^{-\alpha k}}{2\pi} Re\left(\int_{-\infty}^{\infty} \frac{e^{-r(T-t)}e^{\{Tr[A_1(t)\Sigma_0 + A_2(t)\bar{\Sigma}_0] + C(t) + \varepsilon(T-t)\}}}{(\alpha + i\theta)(\alpha + i\theta + 1)} e^{-i\theta k}d\theta\right) \tag{62}$$

With the

$$\varepsilon(\tau) = Tr\left[\log\left(\frac{1}{\sqrt{(H_1(\tau))^{\beta}(I_1(\tau))^{\bar{\beta}}}}\right) - \frac{1}{2}(\beta M^T + \bar{\beta}\bar{M}^T) + r\gamma\right]$$

3.4. Riccati differential equations under Perturbation techniques

This section is devoted to the implementation of the perturbation technique. We resort to this method to approximate the Call Option price, given the absence of a closed-form solution to the system of Riccati differential equations stemming from the non-commutative nature of matrix multiplication. The attainment of an analytical closed-form solution remains a daunting challenge.

Despite the inherent complexity of the procedure, it retains affine properties and can accommodate higher orders in alignment with the conventional perturbation scheme applied to partial differential equations. However, as OdhiamboJo et al.[38] highlight, it tends to need to be clarified beyond the first order.

Our exploration focuses on the Riccati differential equations intrinsic to the double Wishart stochastic volatility model, when solving the equations. We consider a dimension n = 3. Additionally, we account for the two distinct orders in perturbation, symbolized as p and q. The solution for A(τ) is conveyed in the ensuing form:

$$A(\tau) = \sum_{i,j} p^{\frac{i}{2}} q^{\frac{j}{2}} A^{i,j}(\tau)$$

The perturbation-infused differential equations are then meticulously expanded. This is accomplished by aligning coefficients and isolating terms related to p and q. This methodical process sets the stage for deriving the anticipated approximations.

Let $p = M_1$ and $q = M_2$ are small while v_i quantities remain constant. The approximation at order one (*p, q*) and order the two (*p, q*) with these notations.

$$M = \begin{pmatrix} -p & 0 \\ 0 & -q \end{pmatrix} = -p \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - q \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -pM_1 - qM_2$$

Also

$$\bar{M} = \begin{pmatrix} -\bar{p} & 0 \\ 0 & -\bar{q} \end{pmatrix} = -\bar{p} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \bar{q} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -\bar{p}M_1 - \bar{q}M_2$$

with noting that Q as the implied volatility of the volatility

$$Q = \sqrt{p}\nu_1 M_1 + \sqrt{q}\nu_2 M_2, \quad Q^2 = p\nu_1^2 M_1 + q\nu_2^2 M_2$$

while can be rewritten as

$$Q = \sqrt{p}Q_1 + \sqrt{q}Q_2$$

We can denote the Riccati equations as in the new form

$$A_1^0(\tau) = p[-A_1(\tau)M_1 - M_1A_1(\tau) + 2A_1(\tau)Q^2{}_1A_1(\tau)] + q[-A_1(\tau)M_2 - M_2A_1(\tau) + 2A_1(\tau)^{Q2}{}_2A_1(\tau)] \quad (63)$$

and for

$$\lambda_1'(\tau) = \frac{\gamma(\gamma-1)}{2}\mathbb{I}_2 + p(-\lambda_1(\tau)M_1 - M_1\lambda_1(\tau) + 2\lambda_1(\tau)Q_1^2\lambda_1(\tau)) + 2\sqrt{p}\gamma RQ_1\lambda_1(\tau) \\ +2\sqrt{q}\gamma RQ_2\lambda_1(\tau) + q(-\lambda_1(\tau)M_2 - M_2\lambda_1(\tau) + 2\lambda_1(\tau)Q_2^2\lambda_1(\tau)) \quad (64)$$

$$\lambda_2'(\tau) = \frac{\gamma(\gamma-1)}{2}\mathbb{I}_2 + \bar{p}(-\lambda_2(\tau)M_1 - M_1\lambda_2(\tau) + 2\lambda_2(\tau)\bar{Q}_1^2\lambda_2(\tau)) + 2\sqrt{\bar{p}}\gamma\bar{R}\bar{Q}_1\lambda_2(\tau) \\ +2\sqrt{\bar{q}}\gamma\bar{R}\bar{Q}_2\lambda_2(\tau) + \bar{q}(-\lambda_2(\tau)M_2 - M_2\lambda_2(\tau) + 2\lambda_2(\tau)\bar{Q}_2^2\lambda_2(\tau)) \quad (65)$$

This section proceeds to expand the Riccati functions A_1, A_2, λ_1, λ_2, C, and ε by implementing a second-order perturbation on the Riccati equations. This manipulation is executed to extract the formula for pricing a European call option, which is presented as follows:

$$A_k(\tau) = A_k^{0,0}(\tau) + \sqrt{p}A_k^{1,0}(\tau) + \sqrt{q}A_k^{0,1}(\tau) + pA_k^{2,0}(\tau) + qA_k^{0,2}(\tau) + \sqrt{pq}A_k^{1,1}(\tau) + o(max(p,q)), \quad k = 1,2 \quad (66)$$

Let us determine $A^0{}_k{}^0, \ldots, A_k^{1,1}$

Through identifying of the terms in their respective orders we get

$$(A_1^{0,0}(\tau))' = 0, \quad A_1^{0,0}(\tau) \in M_2(\mathbb{C})$$

Since $A_1(0) = iD_1$, whereas

$$D_1 = \begin{pmatrix} d_1 & 0 \\ 0 & d_1 \end{pmatrix}$$

Then we get

$$A_1^{0,0}(0) + \sqrt{p}A_1^{1,0}(0) + \sqrt{q}A_1^{0,1}(0) + pA_1^{2,0}(0) + qA_1^{0,2}(0) + \sqrt{pq}A_1^{1,1}(0) = iD_1 \tag{67}$$

therefore

$$A_1^{0,0}(0) = iD_1 = id_1\mathbb{I}_2$$
$$A_1^{1,0}(0) = A_1^{0,1}(0) = A_1^{2,0}(0) = A_1^{0,2}(0) = A_1^{1,1}(0) = (0)$$
$$A_1^{2,0}(\tau) = -2d_1(i + d_1\nu_1^2)\tau M_1$$
$$A_1^{0,2}(\tau) = -2d_1(i + d_1\nu_2^2)\tau M_2$$

Also

$$(A_1^{1,1}(\tau))' = 0, \quad \Longrightarrow A_1^{1,1}(\tau) = C$$

Since all $A_1^{1,1}(0) = 0$ for values of all $\tau \in$ R+ the $C = 0$.

For function $A_2(\tau)$, we again follow a similar procedure

$$A_2'(\tau) = \bar{p}[-A_2(\tau)M_1 - M_1A_2(\tau) + 2A_2(\tau)\bar{Q}_1^2A_2(\tau)] + \bar{q}[-A_2(\tau)M_2 - M_2A_2(\tau) + 2A_2(\tau)\bar{Q}_2^2A_2(\tau)]$$
$$A_2'(\tau) = (A_2^{0,0}(\tau))' + \sqrt{\bar{p}}(A_2^{1,0}(\tau))' + \sqrt{\bar{q}}(A_2^{0,1}(\tau))' + \bar{p}(A_2^{2,0}(\tau))' + \bar{q}(A_2^{0,2}(\tau))' + \sqrt{\bar{p}\bar{q}}(A_2^{1,1}(\tau))'$$

therefore

$$(A_2^{0,0}(\tau))' = 0, \quad \Longrightarrow A_2^{0,0}(\tau) \in M_2(\mathbb{C})$$

Since the value of $A_2(0) = iD_2$, whereas

$$D_2 = \begin{pmatrix} d_2 & 0 \\ 0 & d_2 \end{pmatrix}$$

then we will get

$$(A_2^{0,0}(0))' + \sqrt{\bar{p}}(A_2^{1,0}(0))' + \sqrt{\bar{q}}(A_2^{0,1}(0))' + \bar{p}(A_2^{2,0}(0))' + \bar{q}(A_2^{0,2}(0))' + \sqrt{\bar{p}\bar{q}}(A_2^{1,1}(0))' = iD_2$$
$$A_2^{0,0}(0) = iD_2 = id_2\mathbb{I}_2 \tag{68}$$
$$A_2^{1,0}(0) = A_2^{0,1}(0) = A_2^{2,0}(0) = A_2^{0,2}(0) = A_2^{1,1}(0) = (0)$$

we proceed through getting

$$A_2^{2,0}(\tau) = -2d_2(i + d_2\nu_1^2)\tau M_1$$
$$A_2^{0,2}(\tau) = -2d_2(i + d_2\bar{\nu}_2^2)\tau M_2$$

and $(A_2^{1,1}(\tau))' = 0, \quad A_2^{1,1}(\tau) = 0$, since $A_2^{1,1}(0) = 0$ for all $\tau \in$ R+, then $C = 0$ constant. Meaning that we obtain similar functions of λ_1 and λ_2 using the given Ordinary differential equations as

$$\begin{aligned}\lambda_1'(\tau) = \frac{\gamma(\gamma-1)}{2}\mathbb{I}_2 + p(-\lambda_1(\tau)M_1 - M_1\lambda_1(\tau) + 2\lambda_1(\tau)Q_1^2\lambda_1(\tau)) + 2\sqrt{p}\gamma RQ_1\lambda_1(\tau) \\ +2\sqrt{q}\gamma RQ_2\lambda_1(\tau) + q(-\lambda_1(\tau)M_2 - M_2\lambda_1(\tau) + 2\lambda_1(\tau)Q_2^2\lambda_1(\tau))\end{aligned} \tag{69}$$

$$\lambda_1(\tau) = \lambda_1^{0,0}(\tau) + \sqrt{p}\lambda_1^{1,0}(\tau) + \sqrt{q}\lambda_1^{0,1}(\tau) + p\lambda_1^{2,0}(\tau) + q\lambda_1^{0,2}(\tau) + \sqrt{pq}\lambda_1^{1,1}(\tau) + o(max(p,q)) \tag{70}$$

then

$$\lambda_1^{0,0}(\tau) = \frac{\gamma(\gamma-1)}{2}\tau\mathbb{I}_2$$

$$\lambda_1^{1,0}(\tau) = \frac{\gamma^2(\gamma-1)}{2}\nu_1\tau^2(RM_1)$$

$$\lambda_1^{0,1}(\tau) = \frac{\gamma^2(\gamma-1)}{2}\nu_2\tau^2(RM_2)$$

also for

$$\lambda_1^{2,0}(\tau) = -\frac{\gamma(\gamma-1)}{2}\tau^2 M_1 + \frac{\gamma^2(\gamma-1)^2}{6}\nu_1^2\tau^3 M_1 + \frac{\gamma^3(\gamma-1)}{3}\nu_1^2\tau^3(RM_1)^2$$

$$\lambda_1^{0,2}(\tau) = -\frac{\gamma(\gamma-1)}{2}\tau^2 M_2 + \frac{\gamma^2(\gamma-1)}{6}\nu_2^2\tau^3 M_2 + \frac{\gamma^3(\gamma-1)}{3}\nu_2^2\tau^3(RM_2)^2$$

$$\lambda_1^{1,1}(\tau) = \frac{\gamma^3(\gamma-1)}{3}\nu_1\nu_2\tau^3[(RM_1)(RM_2)+(RM_2)(RM_1)]$$

Then similarly for λ_2,we have these similar approximations

$$\lambda_2^{0,0}(\tau) = \frac{\gamma(\gamma-1)}{2}\tau\mathbb{I}_2$$

$$\lambda_2^{1,0}(\tau) = \frac{\gamma^2(\gamma-1)}{2}\bar{\nu}_1\tau^2(\bar{R}M_1)$$

$$\lambda_2^{0,1}(\tau) = \frac{\gamma^2(\gamma-1)}{2}\bar{\nu}_2\tau^2(\bar{R}M_2)$$

$$\lambda_2^{2,0}(\tau) = -\frac{\gamma(\gamma-1)}{2}\tau^2 M_1 + \frac{\gamma^2(\gamma-1)^2}{6}\bar{\nu}_1^2\tau^3 M_1 + \frac{\gamma^3(\gamma-1)}{3}\bar{\nu}_1^2\tau^3(\bar{R}M_1)^2$$

$$\lambda_2^{0,2}(\tau) = -\frac{\gamma(\gamma-1)}{2}\tau^2 M_2 + \frac{\gamma^2(\gamma-1)}{6}\bar{\nu}_2^2\tau^3 M_2 + \frac{\gamma^3(\gamma-1)}{3}\bar{\nu}_2^2\tau^3(\bar{R}M_2)^2$$

$$\lambda_2^{1,1}(\tau) = \frac{\gamma^3(\gamma-1)}{3}\bar{\nu}_1\bar{\nu}_2\tau^3[(\bar{R}M_1)(\bar{R}M_2)+(\bar{R}M_2)(\bar{R}M_1)]$$

Therefore we relook at the specific Riccati differential equation for the value of $C(\tau)$

$$C'(\tau) = \beta Tr[p\nu_1^2 M_1 A_1(\tau) + q\nu_2^2 M_2 A_1(\tau)] + \bar{\beta}Tr[p\bar{\nu}_1^2 M_1 A_2(\tau) + q\bar{\nu}_2^2 M_2 A_2(\tau)]$$

$$C^{0,0}(\tau) = 0$$

As the value of $C(0) = 0$

$$C1,0(\tau) = C0,1(\tau) = 0$$

$$C2,0(\tau) = i\beta v12d1\tau + i\beta^{-}v^{-}12d2\tau$$

$$C0,2(\tau) = i\beta v22d1\tau + i\beta^{-}v^{-}22d2\tau$$

with

$$C^{1,1}(\tau) = 0$$

For the constant, ε the differential equation is determined as;

$$\varepsilon'(\tau) = r\gamma + \beta Tr[p\nu_1^2 M_1\lambda_1(\tau) + q\nu_2^2 M_2\lambda_1(\tau)] + \bar{\beta}Tr[p\bar{\nu}_1^2 M_1\lambda_2(\tau) + q\bar{\nu}_2^2 M_2\lambda_2(\tau)]$$

with

$$\varepsilon^{0,0}(\tau) = r\gamma\tau$$

$$\varepsilon^{0,1}(\tau) = 0$$

$$\varepsilon^{2,0}(\tau) = \beta\nu_1^2\frac{\gamma(\gamma-1)}{4}\tau^2 + \bar{\beta}\bar{\nu}_1^2\frac{\gamma(\gamma-1)}{4}\tau^2$$

$$\varepsilon^{0,2}(\tau) = \beta\nu_2^2\frac{\gamma(\gamma-1)}{4}\tau^2 + \bar{\beta}\bar{\nu}_2^2\frac{\gamma(\gamma-1)}{4}\tau^2$$

$$\varepsilon^{1,1}(\tau) = 0$$

By substituting it back in the derived perturbation function, we get

$$A_1(\tau) = A_1^{0,0}(\tau) + \sqrt{p}A_1^{1,0}(\tau) + \sqrt{q}A_1^{0,1}(\tau) + pA_1^{2,0}(\tau) + qA_1^{0,2}(\tau) + \sqrt{pq}A_1^{1,1}(\tau) + o(max(p,q)) \quad (71)$$

$$A_1(\tau) = \begin{pmatrix} id_1 & 0 \\ 0 & id_1 \end{pmatrix} + \begin{pmatrix} -2pd_1(i + d_1\nu_1^2)\tau & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -2qd_1(i + d_1\nu_2^2)\tau \end{pmatrix} + o(max(p,q))$$

$$A_1(\tau) = \begin{pmatrix} -2pd_1^2\nu_1^2\tau + id_1(1 - 2p\tau) & 0 \\ 0 & -2qd_1^2\nu_2^2\tau + id_1(1 - 2q\tau) \end{pmatrix}$$

and

$$A_2(\tau) = \begin{pmatrix} -2pd_2^2\bar{\nu}_1^2\tau + id_2(1 - 2p\tau) & 0 \\ 0 & -2qd_2^2\bar{\nu}_2^2\tau + id_2(1 - 2q\tau) \end{pmatrix}$$

The determined variance processes denoted as Σ0 and Σ¯0 are as follows

$$A_2(\tau)\bar{\Sigma}_0 = \begin{pmatrix} -2\bar{u}pd_2^2\bar{\nu}_1^2\tau + id_2(1 - 2p\tau)\bar{u} & -2p\bar{v}d_2^2\bar{\nu}_1^2\tau + id_2(1 - 2p\tau)\bar{v} \\ -2q\bar{v}d_2^2\bar{\nu}_2^2\tau + id_2(1 - 2q\tau)\bar{v} & -2q\bar{w}d_2^2\bar{\nu}_2^2\tau + id_2(1 - 2q\tau)\bar{w} \end{pmatrix}$$

getting the trace, and

$$Tr(A_1(\tau)\Sigma_0) = -2d_1^2[up\nu_1^2 + wq\nu_2^2]\tau + i[(1 - 2p\tau)u + (1 - 2q\tau)w]d_1$$

$$Tr(A_2(\tau)\bar{\Sigma}_0) = -2d_2^2[\bar{u}p\bar{\nu}_1^2 + \bar{w}q\bar{\nu}_2^2]\tau + i[(1 - 2p\tau)\bar{u} + (1 - 2q\tau)\bar{w}]d_2$$

For functions C and ε

$$C(\tau) = pC^{2,0}(\tau) + qC^{0,2}(\tau) + o(max(p, q))$$
$$= p(i\beta v_1^2 d_1\tau + i\bar{\beta}\bar{v}_1^2 d_2\tau) + q(i\beta v_2^2 d_1\tau + i\bar{\beta}\bar{v}_2^2 d_2\tau) + o(max(p, q)) = i\beta d_1(pv_1^2 + qv_2^2)\tau + i\beta\bar{d}_2(p\bar{v}_1^2 + q\bar{v}_2^2)\tau + o(max(p, q)) \quad (72)$$

$$\varepsilon(\tau) = \varepsilon^{0,0}(\tau) + p\varepsilon^{2,0}(\tau) + q\varepsilon^{0,2}(\tau) + o(max(p,q))$$
$$= r\gamma\tau + \frac{\gamma(\gamma-1)}{4}\beta(p\nu_1^2 + q\nu_2^2)\tau^2 + \frac{\gamma(\gamma-1)}{4}\bar{\beta}(p\bar{\nu}_1^2 + q\bar{\nu}_2^2)\tau^2 + o(max(p,q)) \quad (73)$$

When you substitute them back in equation (61), we get

$$\begin{aligned}\tilde{\phi}_{\Sigma\bar{\Sigma}}(\tau) = exp\{r\gamma(T-t) + p[-2d_1^2u\nu_1^2t - 2d_2^2\bar{u}\bar{\nu}_1^2t + \frac{\gamma(\gamma-1)}{4}\beta\nu_1^2(T-t)^2 + \frac{\gamma(\gamma-1)}{4}\bar{\beta}\bar{\nu}_1^2(T-t)^2] \\ +q[-2d_1^2w\nu_2^2t - 2d_2^2\bar{w}\bar{\nu}_2^2t + \frac{\gamma(\gamma-1)}{4}\beta\nu_2^2(T-t)^2 + \frac{\gamma(\gamma-1)}{4}\bar{\beta}\bar{\nu}_2^2(T-t)^2]\} \\ \times exp\{i[(1-2pt)ud_1 + (1-2qt)wd_1 + (1-2pt)\bar{u}d_2 \\ +(1-2qt)\bar{w}d_2 + \beta d_1(p\nu_1^2 + q\nu_2^2)t + \bar{\beta}d_2(p\bar{\nu}_1^2 + q\bar{\nu}_2^2)t] + o(max(p,q))\}\end{aligned} \quad (74)$$

The equation can be rewritten as

$$\tilde{\varphi}_{\Sigma\bar{\Sigma}}(\tau) = e\Delta^1(t)ei\Delta^2(t)$$

with

$$\begin{aligned}\Delta_1(t) = \{r\gamma(T-t) + p[-2d_1^2u\nu_1^2t - 2d_2^2\bar{u}\bar{\nu}_1^2t + \frac{\gamma(\gamma-1)}{4}\beta\nu_1^2(T-t)^2 + \frac{\gamma(\gamma-1)}{4}\bar{\beta}\bar{\nu}_1^2(T-t)^2] \\ +q[-2d_1^2w\nu_2^2t - 2d_2^2\bar{w}\bar{\nu}_2^2t + \frac{\gamma(\gamma-1)}{4}\beta\nu_2^2(T-t)^2 + \frac{\gamma(\gamma-1)}{4}\bar{\beta}\bar{\nu}_2^2(T-t)^2]\}\end{aligned}$$

and

$$\Delta_2(t) = \{(1 - 2pt)ud_1 + (1 - 2qt)wd_1 + (1 - 2pt)\bar{u}d_2 + (1 - 2qt)\bar{w}d_2 + \beta d_1(pv_1^2 + qv_2^2)t + \beta\bar{d}_2(p\bar{v}_1^2 + q\bar{v}_2^2)t + o(max(p, q))\}$$

From equation (69)

$$C_t(T,K) = \frac{e^{-\alpha k}}{2\pi} Re\left(\int_{-\infty}^{\infty} e^{-r(T-t)}\tilde{\phi}_{\Sigma\bar{\Sigma}}(t)\phi(\theta)e^{-i\theta k}d\theta\right)$$
$$\approx \frac{e^{-\alpha k}}{2\pi} Re(\int_{-\infty}^{\infty} \frac{e^{-r(T-t)}e^{\Delta_1(t)}e^{i\Delta_2(t)}e^{-i\theta k}}{(\alpha+i\theta)(\alpha+1+i\theta)}d\theta)$$

since

$$\int_{-\infty}^{\infty} \frac{e^{-r(T-t)}e^{\Delta_1(t)}e^{i\Delta_2(t)}e^{-i\theta k}}{(\alpha+i\theta)(\alpha+1+i\theta)}d\theta = e^{-r(T-t)}e^{\Delta_1(t)}e^{i\Delta_2(t)}\int_{-\infty}^{\infty} \frac{e^{-i\theta k}}{(\alpha+i\theta)(\alpha+1+i\theta)}d\theta$$
$$= e^{-r(T-t)}e^{\Delta_1(t)}e^{i\Delta_2(t)}[2\pi(e^{\alpha k} - e^{k(\alpha+1)})]$$

In the end, we get the approximated value of the European option call value as

$$Ct(T,K) \approx e(-r(T-t)+\Delta 1(t))(1 - ek)cos\Delta 2(t) \quad (75)$$

4. Numerical analysis

The volatility specification of the generalized Wishart model lends remarkable flexibility, enabling it to generate price predictions that closely mimic actual market behavior. We derive our market data from QQQ options, a fund managed by Invesco that mirrors the performance of the stocks listed under the S & P 500 Index, covering the period from April 2020 to April 2022.

Based on the selected parameter values, it's noticeable that the price predictions generated by the double Wishart model align closely with the market price over short maturity periods. This alignment is contingent on the chosen model parameters, specifically, $\gamma = 0.6, \beta = 3, d_1 = 0.5, d_2 = 0.55$. This instance illustrates the superior flexibility offered by the double Wishart volatility model as shown in **Figure 1**.

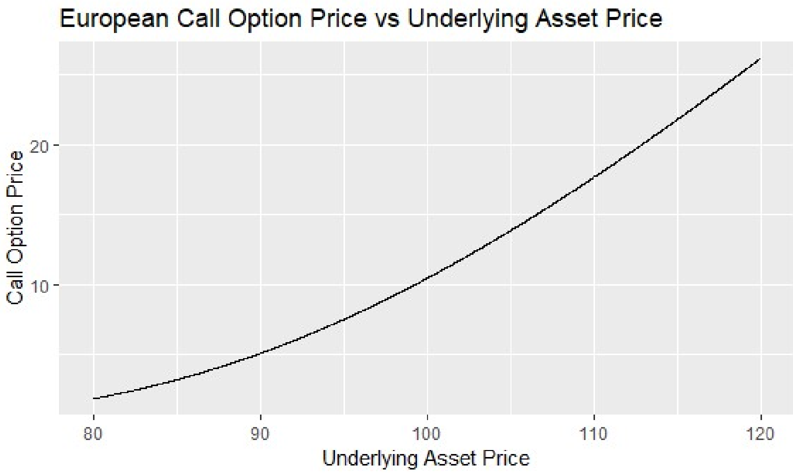

Figure 1. European call option Vs S & P 500 Index Price in 4. months.

The model successfully mirrors market price trends in longer maturity periods of a year. It's noteworthy that the parameters within the model play a pivotal role in shaping the price trajectories, providing significant insights to those holding long positions, and preventing the occurrence of unwarranted arbitrage profits.

Our analysis revealed a significant relationship between the forecasted Call Option prices and the Market Index. The Call Option prices and the Market Index displayed a similar movement pattern. When the Market Index experienced an increase, there was a corresponding increase in the Call Option prices and vice versa. This observation suggests that the movement strongly influenced the Call Option prices in the Market Index.

However, it was also observed that the volatility of the Call Option prices was slightly higher than that of the Market Index. It indicates that while the Call Option prices generally followed the trend of the Market

Index, they were subject to more significant fluctuations. This higher volatility in Call Option prices can be attributed to the inherent risk associated with options trading as shown in **Figure 2**.

Moreover, the Call Option prices demonstrated a delayed reaction to changes in the Market Index. When there was a significant movement in the Market Index, it took a while for the Call Option prices to reflect this change.

In conclusion, our findings suggest that the forecasted Call Option prices strongly correlate with the Market Index such as S & P 500 thus proving that our proposed model does better than the market index. Still, they are also subject to higher volatility and a delayed response to market changes as shown in Figure 2. Future work should involve further analysis of these observations and the development of more sophisticated forecasting models to accurately predict Call Option prices based on movements in the Market Index.

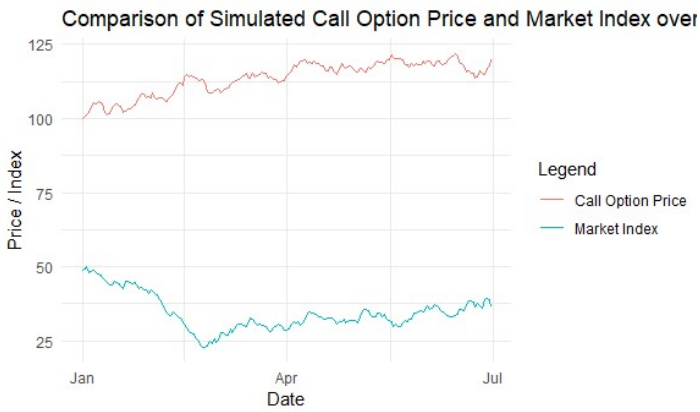

Figure 2. A comparison between Forecasted Call option Vs Market Index.

5. Conclusion

We can efficiently address European call options pricing by expanding the Heston model into a multiple-factor structure involving dual dependency matrices. This strategic approach enhances the fitting accuracy of financial market data across both short and long-term maturities. Our proposed Wishart affine model retains the advantage of analytical tractability, paving the way for a closed-form solution of the conditional characteristic function. This function is pivotal in articulating the price of a call option, achievable through applying Fourier transformations and perturbation techniques.

A noteworthy feature of our model is its flexibility, driven by the effects of its parameters. This flexibility makes it an excellent fit for relevant data across various maturities, from short to long-term. In terms of future research directions, the precise simulation with discretization schemes is a worthy area to delve into. Additionally, investigating the behaviour of non-diagonal matrix components within our model would provide valuable insights. Regarding room for further research, Numerical Analysis and analytical consequences always done using data to test the proposed model will be done in the following paper as this study was more on analytical proof.

Data availability

The data used in this paper is available in Microsoft Excel Worksheet and Python programming language. Upon request, the data will be availed for anyone who needs it for now and as well as in the future.

Conflict of interest

The author declares no conflict of interest.

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Appendix

The correlation structures

The Wiener process matrices W_t, Z_t are correlated to result to a specific constant correlated matrix $R \in M_n$, in Kang[11], describing the correlation structure for Z_t

$$Zt := WtRT + Bt\overline{\mathrm{pI} - RRT}$$

whereas I is identity matrix, T is transpose, and B_t is an independent Wiener Process matrix from W_t. The correlation structure is a Wiener Process *Proof.* Z_t is matrix Wiener Process iff $a,b \in \mathrm{R}^n$

$$Cov_t(dZ_ta,\ dZ_tb) = \mathrm{E}t[(dZ_ta)(dZ_tb)^T] = a^T bIdt$$

Since the value of

$$Cov_t(dZ_ta, dZ_tb) = \mathbb{E}_t[(dW_tR^Ta + dB_t\sqrt{I - RR^Ta})(dW_tR^Tb + dB_t\sqrt{I - RR^Tb})]$$

$$= Cov_t(dW_tR^Ta, dW_tR^Tb) + Cov_t(dB_t\sqrt{I - RR^Ta}, dB_t\sqrt{I - RR^Tb})$$

$$= a^TRR^TbIdt + a^T(I - RR^T)bIdt$$

$$Cov_t(dZ_ta, dZ_tb) = a^TbIdt$$

Generalized Wishart Processes and the characteristic functions

Let D be symmetric matrix, the given conditional characteristic function of the generalized Wishart Σ_t and $\bar{\Sigma}_t$ is derived by

$$\begin{aligned}\tilde{\phi}^{D_1D_2}_{\Sigma_t,\bar{\Sigma}_t} &= \mathbb{E}\left[exp\{iTr[D_1\Sigma_t + D_2\bar{\Sigma}_t]\}\right] \\ &= exp\{Tr[A_1(\tau)\Sigma_t + A_2(\tau)\bar{\Sigma}_t] + C(\tau)\}\end{aligned}$$

whereas $A_1(\tau),A_2(\tau) \in M_n$ and $C \in \mathrm{C}$ are used in the verification the given dynamics

Proof. Getting expressions of $A_1(\tau),A_2(\tau)$ and $C(\tau)$, using the Riccati equations

$$\begin{cases}\frac{dA_1(\tau)}{d\tau} = A_1(\tau)M + M^TA_1(\tau) + 2A_1(\tau)Q^TQA_1(\tau) \\ A_1(0) = iD_1\end{cases} \tag{76}$$

$$\begin{cases}\frac{dA_2(\tau)}{d\tau} &= A_2(\tau)\bar{M} + \bar{M}^TA_2(\tau) + 2A_2(\tau)\bar{Q}^T\bar{Q}A_2(\tau) \\ A_2(0) &= iD_2\end{cases}$$

$$\begin{cases}\frac{dC(\tau)}{d\tau} &= \beta Tr[Q^TQA_1(\tau)] + \bar{\beta}Tr[\bar{Q}^T\bar{Q}A_2(\tau)] \\ C(0) &= 0\end{cases}$$

Now

$$A_1(\tau) = h_1^{-1}(\tau)h_2(\tau)$$

$$\frac{d}{d\tau}\begin{pmatrix}h_2(\tau) & h_1(\tau)\end{pmatrix} = \begin{pmatrix}h_2(\tau) & h_1(\tau)\end{pmatrix}\begin{pmatrix}M & -2Q^TQ \\ 0 & -M^T\end{pmatrix}$$

$$A_2(\tau) = i_1^{-1}(\tau)i_2(\tau)$$

$$\frac{d}{d\tau}\begin{pmatrix}i_2(\tau) & i_1(\tau)\end{pmatrix} = \begin{pmatrix}i_2(\tau) & i_1(\tau)\end{pmatrix}\begin{pmatrix}\bar{M} & -2\bar{Q}^T\bar{Q} \\ 0 & -\bar{M}^T\end{pmatrix}$$

We get

$$\begin{pmatrix}i_2(\tau) & i_1(\tau)\end{pmatrix} = \begin{pmatrix}iD_2A_2^{11}(\tau) + A_2^{21}(\tau) & iD_2A_2^{12}(\tau) + A_2^{22}(\tau)\end{pmatrix}$$

For

$$A_1(\tau) = h_1^{-1}(\tau)h_2(\tau)$$

$$h_1(\tau) = iD_1A_1^{12}(\tau) + A_1^{22}(\tau)$$

$$h_2(\tau) = iD_1A^{11}{}_1(\tau) + A^{21}{}_1(\tau)$$

$$A1(\tau) = (iD1A121\ (\tau) + A221\ (\tau))-1(iD1A111\ (\tau) + A121\ (\tau))$$

$$A_2(\tau) = i^{-}{}_1{}^1(\tau)i_2(\tau)\ i_1(\tau) = iD_2A^{12}{}_2(\tau) + A^{22}{}_2(\tau)\ i_2(\tau) = iD_2A^{11}{}_2(\tau) + A^{21}{}_2(\tau)$$

$$A2(\tau) = (iD2A122\ (\tau) + A222\ (\tau))-1(iD2A112\ (\tau) + A221\ (\tau))$$