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Commentary

Note on "stability and data dependence results for jungck-type iteration scheme"

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Abstract: This note reviews the iterative methods introduced in "Stability and Data Dependence Results for Jungck-Type Iteration Scheme". It has been observed that these methods are not entirely new to a significant extent, as they bear resemblance to previously established approaches. Additionally, the primary iterative method proposed in the paper lacks efficiency, particularly when compared to more advanced or well-known methods. As a result, while the methods may offer some value, they do not represent a significant breakthrough in iterative techniques.

Keywords: algorithms; iteration methods; convergence order

Introduction and main results

Solving the nonlinear equation

$$f(x) = 0,\ x \in \mathbb{R}$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function and D an open interval, is one of the oldest problems in numerical analysis [1,2].

We know that one of the fundamental algorithm for solving nonlinear equations is so-called fixed point iteration method. In the fixed-point iteration method for solving nonlinear Equation (1), the equation is usually rewritten as

$$x = \breve{g}(x)$$

where

(i) there exists $[a, b]$ such that $\breve{g}(x) \in [a, b]$ for all $x \in [a, b]$,

(ii) there exists $[a, b]$ such that $|\breve{g}^{'}(x)| \leq L < 1$ for all $x \in [a, b]$.

Considering the following iteration scheme

$$x_{n+1} = \breve{g}(x_n),\ n = 0, 1, 2 \dots$$

and starting with a suitable initial approximation x_0, we built up a sequence of approximations, say $\{x_n\}$, for the solution of nonlinear equation, say $\check{T}$. The scheme will be converge to $\check{T}$, provided that

(i) the initial approximation x_0 is chosen in the interval $[a, b]$,

(ii) $|\breve{g}^{'}(x)| < 1$ for all $x \in [a, b]$,

(iii) $a \leq \breve{g}(x) \leq b$ for all $x \in [a, b]$.

Definition 1. [1,2] *Let* $\{x_n\}$ *converges to* ν. *If there exist an integer* p *and a real positive* *constant C such that*

$$\lim_{n\to\infty}\frac{|x_{n+1}-\nu|}{(x_n-\nu)^{\rho}}=C$$

then ρ is called the order of convergence. The efficiency index of an iterative method is a metric used to compare different iterative methods. It is defined as EI=$\rho^{\frac{1}{\lambda}}$, where ρ is the local order of convergence of the method and λ is the number of function evaluations needed to carry out the method per iteration.

To determine the order of convergence of the sequence $\{x_n\}$, let us consider the Taylor expansion of $\breve{g}(x_n)$

$$\breve{g}(x_n)=\breve{g}(x)+\frac{\breve{g}^{'}(x)}{1!}(x_n-x)+\frac{\breve{g}^{''}(x)}{2!}(x_n-x)^2+...+\frac{\breve{g}^{k}(x)}{k!}(x_n-x)^k+...$$

We have

$$x_{n+1}-x=\frac{\breve{g}^{'}(x)}{1!}(x_n-x)+\frac{\breve{g}^{''}(x)}{2!}(x_n-x)^2+...+\frac{\breve{g}^{k}(x)}{k!}(x_n-x)^k+...$$

Theorem 1. $[1,2]$ *Suppose that $\breve{g}\in C^n[\tilde{a},b]$. If $\breve{g}^k(x)=0$, for $k=1,2,...,p-1$ and $\breve{g}^k(x)\neq 0$, then the sequence $\{x_n\}$ has p as its order of convergence.*

Remark 1. *It is well known that the fixed point method has first order of convergence.*

Let the nonlinear Equation (1) has a simple root ζ or equivalently ζ be the coincidence point of S and T (i.e, $S\zeta=T\zeta$), where $S,T:D\subset\mathbb{R}\to\mathbb{R}$; $T(D)\subset S(D)$, S is onto, and T is sufficiently differentiable in the neighborhood of ζ. Let $x_0\in D$ be the initial guess near to ζ. The Equation (1) can be written as

$$Sx=Tx$$

In order to convey the idea, some details from [2] are as follows. We can modify (2) by multiplying $\tau\neq -1$ on both sides as follows:

$$Sx+\tau Sx=\tau Sx+Tx$$

implies

$$\frac{\tau Sx+Tx}{\tau+1}=S_{\tau}x \text{ (say)} \tag{1}$$

where τ is an arbitrary number. In order (3) to be efficient, we can choose $\tau=-T^{'}x$ such that

$$S_{\tau}x=\frac{\tau Sx+Tx}{\tau+1}=\frac{-T^{'}xSx+Tx}{1-T^{'}x},\ \tau=-T^{'}x \tag{2}$$

This formation allows us to suggest the following iteration scheme:

For a given x_0, we calculate the the approximation solution x_{n+1}, by the iteration scheme [2]:

$$Sx_{n+1}=\frac{\tau Sx_n+Tx_n}{\tau+1}=\frac{-T^{'}x_nSx_n+Tx_n}{1-T^{'}x_n};\ \tau=-T^{'}x \tag{3}$$

Remark 2. *1. The algorithm (MJM) takes the form,*

$$\begin{aligned} Sx_{n+1} &= \frac{\tau Sx_n + Tx_n}{\tau+1}S \\ &= \frac{\tau}{\tau+1}Sx_n + \frac{1}{\tau+1}Tx_n \\ &= (1-\theta)Sx_n + \theta Tx_n; \theta = \frac{1}{\tau+1} \in (0,1] \end{aligned} \tag{4}$$

which is due to Singh et al., [3]. Thus the algorithms (MJM) and (S) are equivalent.

2. For $Sv = v$, the algorithm (MJM) yields,

$$x_{n+1} = \frac{\tau x_n + Tx_n}{1+\tau} = \frac{-T^{'}x_n x_n + Tx_n}{1-T^{'}x_n}, \tau = -T^{'}x_n \neq -1, \tag{5}$$

which is due to Kang et al. [1].

3. The algorithm (A) can be rewritten as,

$$x_{n+1} = (1-\theta)x_n + \theta Tx_n; \theta = \frac{1}{\tau+1} \in (0,1]$$

which is due to Kirik [4] and Mann [5] respectively.

Let $(E, ||.||)$ be an arbitrary Banach space and $Y \subset E$. Suppose that $S, T : Y \longrightarrow E$ are two non-self mappings such that $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of E and S is onto.

The following algorithm is due to Sharma P., et al. [6]:

$$\begin{aligned} x_0 &\in Y, B \\ Sz_n &= Tx_n, \\ Sy_n &= Tz_n, \\ Sx_{n+1} &= \frac{mSy_n + Ty_n}{1+m}, \\ m &> 0 \text{ is a real number}, n = 0, 1, 2, ... \end{aligned} \tag{6}$$

which is the combination of extended (MJM) with two-step composition of Jungck iteration method.

We comment as follows:

Remark 3. *1. The algorithm (7) in [6] in not new and is actually due to Kang et al. [1]. Also the idea of corrector-step of algorithm (B) is extracted from (MJM) [2].*

2. The order of convergence of algorithm (A) is two with efficiency index $2^{\frac{1}{2}}$ [1]. Thus the results of Theorem 2 ([22] of [6] are coincides with the proof of convergence of algorithm (A) for $m = -T^{'}x_n$, consequently Theorem 2 is not new.

3. The claim about $m > 0$ is not always true as $m = -T^{'}x_n$. For the implementation of (MJM) or (B), it is better to proceed with the convergence criteria of usual Jungck iteration method [7].

4. From the corrector-step of algorithm (B),

$$\begin{aligned} Sx_{n+1} &= \frac{mSy_n + Ty_n}{m+1} \\ &= \frac{m}{m+1}Sy_n + \frac{1}{m+1}Ty_n \\ &= (1-\theta)Sy_n + \theta Ty_n; \theta = \frac{1}{m+1} \in (0,1] \end{aligned}$$

and the algorithm (B) takes the form:

$$\begin{aligned} x_0 &\in Y, BJ \\ Sz_n &= Tx_n, \\ Sy_n &= Tz_n, \\ Sx_{n+1} &= (1-\theta)Sy_n + \theta Ty_n; \theta = \frac{1}{m+1} \in (0,1] \\ n &= 0, 1, 2, ... \end{aligned} \tag{7}$$

5.The convergence order of algorithm (MJM) or (S) is two with efficiency index $2^{\frac{1}{3}}$*. The convergence order of algorithm (B) or (BJ) is two with efficiency index* $2^{\frac{1}{7}} < 2^{\frac{1}{3}}$*.*

6. The performance of the iteration methods (MJM), (B) or (BJ) can be easily checked through a simple code written in MATLAB or MAPLE for the examples provided in [6].

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