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Article

Legendre-Galerkin spectral algorithm for fractional-order BVPs: Application to the Bagley-Torvik equation

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Abstract: Herein, we provide an efficient spectral Galerkin algorithm, according to a special type of shifted Legendre basis for finding a semi-analytic solution to the Liouville-Caputo fractional boundary value problem. The algorithm's main goal is to transform the fractional differential problem into a linear system with efficiently invertible, well-structured matrices. The convergence rates of the algorithm are carefully obtained as well as the error bound.

Keywords: Legendre polynomials; Galerkin method; fractional differential equations; convergence analysis
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1. Introduction

The fractional calculus (FC) began almost three centuries ago. Recently, FC has played a vital part in a variety of scientific fields. FC has been considered one of the most effective methods for describing long-memory processes. Mathematicians, as well as physicists and engineers, are interested in fractional-order models. Research in fractional calculus has developed over the past ten years as it has been shown to give the necessary tools for diffusion and simulating anomalous transport. These models may explain granular and porous flows, transport in fusion plasmas, and biological activities. We take the motion of a nanoparticle in the air as an example, which is stochastic and challenging to model using a traditional differential equation. On the other hand, if the air is viewed as a fractal space on the scale of a molecule, then the nanoparticle's motion is predictable and may be described using a fractal-fractional model. Therefore, for a porous media, we need two scales: one substantial enough to maintain the continuum hypothesis and one narrow sufficient to measure porosity [1]. Another example seems to be the Moon's periodic motion, but it becomes stochastic when measured at very long distances; in this case, the Heisenberg-like uncertainty principle applies [2].

Legendre and shifted Legendre polynomials have developed as an extremely effective approximation for numerical analysis with orthogonality restrictions. For the effectiveness of these polynomials' accuracy, authors used Legendre polynomials in many of their research like these papers [3–5].

Spectral methods are becoming increasingly important in a variety of applied sciences. These methods approximate the solution of a fractional differential equation, that's frequently orthogonal, using a linear combination of specific polynomials. When in comparison with finite-difference techniques, these methods offer numerous benefits. While spectral methods are worldwide, finite-difference methods can be obtained by employing a local polynomial interpolant to approximate a function and its derivatives. When several decimal places of accuracy are needed for solutions in chemical and physical applications, it is desirable to use spectral methods since they can yield an exponential convergence of the results. The studies [6–10] for a few significant uses of spectral techniques.

Bagley-Torvik equation finds wide-ranging applications in science and engineering fields [11–14]. The physical implications of the Bagley-Torvik equation for describing viscoelastic materials and simulating the fluid mechanics mechanism have drawn a great deal of interest from researchers. An abundance of effective methods has been devised to solve the Bagley-Torvik problem. For instance, the analytic approach for a generalized Bagley-Torvik problem through the Prabhakar and Wiman functions was explored in [15], along with a discussion of the solution's existence and uniqueness findings. The Adomian decomposition technique for a Bagley-Torvik equation was employed by the authors in [16]. Also, for the numerical solutions of a Bagley-Torvik equation, the wavelet, generalized Bessel polynomial, and Galerkin Gegenbauer expansion in conjunction with operational matrices were used in [17–21].

In this research, we are interested in finding a semi-analytic solution for the Liouville-Caputo fractional Bagley-Torvik differential problem:

$$\upsilon''(\varepsilon) + \rho\, D^{\gamma}\, \upsilon(\varepsilon) + \sigma\, \upsilon(\varepsilon) = g(\varepsilon), 0 \leqslant \varepsilon \leqslant 1, 1 < \gamma < 2, \tag{1}$$

where ρ and σ are real constants and $g(\varepsilon)$ is given continuous function. Applying the following non-homogeneous boundary conditions

$$\upsilon(0) = \alpha, \upsilon(1) = \beta. \tag{2}$$

The general structure of the article is as next: In Sect. 2, Liouville-Caputo fractional calculus is discussed in depth along with their fundamental relations. Basic relations for shifted Legendre polynomials have existed in Sect. 3. Section 4 describes how we created the basis for the shifted Legendre polynomials. The operational matrix with shifted Legendre polynomials, found in Sect. 5, is used to solve fractional Bagley-Torvik problems. Sect. 6 investigates the shifted Legendre expansion's error evaluation. Section 7 provides three numerical examples to demonstrate the suggested method's effectiveness and applicability. In Section 8, we introduce some concluding observations.

2. Some essential properties of Liouville-Caputo fractional calculus

The objective of this section is to provide some background information and essential definitions for Liouville-Caputo fractional calculus theory.

Definition 1. *In any function* $\zeta(\varepsilon)$ *specified on* $[a, b]$, $\gamma > 0$, $n = \lceil\gamma\rceil$, *and* $\varepsilon > 0$. *The following are the definitions of the right and left-handed Liouville-Caputo fractional order derivatives:*

$${}_{a}^{C}D_{\varepsilon}^{\gamma}\zeta(\varepsilon) = \frac{1}{\Gamma(n-\gamma)}\int_{a}^{\varepsilon}\frac{\zeta^{(n)}(\tau)}{(\varepsilon-\tau)^{1-n+\gamma}}\,d\tau, \tag{3}$$

$${}_{\varepsilon}^{C}D_{b}^{\gamma}\,\zeta(\varepsilon) = \frac{1}{\Gamma(n-\gamma)}\int_{\varepsilon}^{b}\frac{(-1)^{n}\zeta^{(n)}(\tau)}{(\tau-\varepsilon)^{1-n+\gamma}}\,d\tau, \tag{4}$$

where $n \in \mathbb{N}$.

Not to be overlooked, the operator ${}^{C}D^{\gamma}$ meets the following basic properties:

$$ {}^{C}D^{\gamma}c = {}^{C}D^{\gamma}\varepsilon = 0, c{:}\,constant, \tag{5}$$

$$ {}^{C}D^{\gamma}I^{\gamma}\zeta(\varepsilon) = \zeta(\varepsilon), \tag{6}$$

$$ {}^{C}D^{\gamma}\varepsilon^{q} = \frac{\Gamma(q+1)}{\Gamma(q-\gamma+1)}\,\varepsilon^{q-\gamma}, q \in \mathbb{N}, q \geq \lceil\gamma\rceil, \tag{7}$$

$$ I^{\gamma}\,{}^{C}D^{\gamma}\zeta(\varepsilon) = \zeta(\varepsilon) - \sum_{q=0}^{m-1}\frac{\zeta^{(q)}(0^{+})}{q!}(\varepsilon-a)^{q}, \varepsilon > 0. \tag{8}$$

For several more extra properties of fractional integrals and derivatives, see [22].

In the Fractional Bagley-Torvik differential Equations (1) and (2), the non-homogeneous boundary conditions are converted to homogeneous using the following relation [23]

$$ \upsilon(\varepsilon) = u(\varepsilon) + \alpha(1-\varepsilon) + \beta\varepsilon, \tag{9}$$

the second derivative of $\upsilon(\varepsilon)$ is

$$ \upsilon''(\varepsilon) = u''(\varepsilon), \tag{10}$$

and the fractional of order γ Liouville-Caputo derivative $\upsilon(\varepsilon)$ is

$$ D^{\gamma}\upsilon(\varepsilon) \quad = D^{\gamma}u(\varepsilon), \tag{11}$$

we substitute (10) and (11) in (1), then an Equation (1) and non-homogeneous boundary conditions (9) become as the following

$$ u''(\varepsilon) + \rho D^{\gamma}u(\varepsilon) + \sigma u(\varepsilon) = f(\varepsilon), u(0) = 0, u(1) = 0, \tag{12}$$

where $f(\varepsilon)$ is provided by

$$ f(\varepsilon) = g(\varepsilon) - \sigma(\alpha(1-\varepsilon) + \beta\varepsilon). \tag{13}$$

3. Fundamental properties of the shifted Legendre polynomials

The standard Legendre polynomials $\omega_k(\varepsilon)$; $-1 \leq \varepsilon \leq 1$ create a full orthogonal system for $L_w^2[-1,1]$ with the weight function $w(\varepsilon) = 1$. Furthermore, these polynomials meet the following orthogonality condition

$$ \int_{-1}^{1}\omega_m(\varepsilon)\omega_n(\varepsilon)d\varepsilon = \begin{cases} 0, & m \neq n, \\ \dfrac{2}{1+2n}, & m = n. \end{cases} \tag{14}$$

Rodrigues' formula Legendre polynomials is supplied by

$$ \omega_n(\varepsilon) = \frac{2^{-n}}{n!}\frac{d^n}{d\,\varepsilon^n}(\varepsilon^2-1)^n, \tag{15}$$

where $\psi_k(\varepsilon)$ refers to shifted Legendre polynomials formed on the interval $[0,1]$ as:

$$ \psi_k(\varepsilon) = \omega_k(2\varepsilon - 1). \tag{16}$$

The shifted Legendre polynomials create a full orthogonal system for $L^2[0,1]$. They display a subsequent orthogonality relation

$$ \int_{0}^{1}\psi_i(\varepsilon)\psi_k(\varepsilon)d\varepsilon = \begin{cases} 0, & k \neq i, \\ \dfrac{1}{1+2k}, & k = i, \end{cases} \tag{17}$$

and they have an analytic formulation (see to [24])

$$ \psi_k(\varepsilon) = \sum_{i=0}^{k}\frac{(-1)^{i+k}(k+i)!}{(i!)^2(k-i)!}\varepsilon^i. \tag{18}$$

For a function $g(\varepsilon) \in L^2[0,1]$, $g(\varepsilon)$ could be expanded in terms of the shifted Legendre basis that follows:

$$ g(\varepsilon) = \textstyle\sum_{k=0}^{\infty} a_k\psi_k(\varepsilon), \tag{19}$$

where,

$$a_k = (2k+1)\int_0^1 g(\varepsilon)\psi_k(\varepsilon)d\varepsilon. \tag{20}$$

Typically, just the first $(M+1)$-terms with the shifted Legendre polynomials are looked at. So, we have

$$g(\varepsilon) \approx g_M(\varepsilon) = \sum_{k=0}^{M} a_k\, \psi_k(\varepsilon). \tag{21}$$

For more properties of Legendre polynomials see, for example [24].

4. Choice of basis

This section chooses a set of orthogonal polynomials that satisfies non-homogeneous boundary conditions (2). We create the next approximate solution $\upsilon_M(\varepsilon)$

$$\upsilon(\varepsilon) \approx \upsilon_M(\varepsilon) = \sum_{k=0}^{M} a_k \phi_k(\varepsilon);\ a = [a_0, a_1, \dots, a_M]^T$$

and

$$\boldsymbol{\phi}(\varepsilon) = [\phi_0(\varepsilon), \phi_1(\varepsilon), \dots, \phi_M(\varepsilon)]^T, \tag{22}$$

where,

$$\phi_k(\varepsilon) = \psi_{k+2}(\varepsilon) - \psi_k(\varepsilon); k = 0,1,\cdots M. \tag{23}$$

Since $\psi_k(0) = (-1)^k$ and $\psi_k(1) = 1$, we conclude that

$$\phi_k(0) = \phi_k(1) = 0. \tag{24}$$

In the study [25], the authors recommended the basis (23), this basis is orthogonal with the wight function $w(\varepsilon) = (\varepsilon - \varepsilon^2)^{-1}$, in other words,

$$\int_0^1 \phi_i(\varepsilon)\, \phi_j(\varepsilon)\, w(\varepsilon)\, d\varepsilon = \begin{cases} 0, & j \neq i, \\ \dfrac{4(2i+3)}{(i+1)(i+2)}, & j = i. \end{cases} \tag{25}$$

5. Legendre-Galerkin method for solving Liouville-Caputo fractional BVPs

For the weight function $w(\varepsilon) = (\varepsilon - \varepsilon^2)^{-1}$, let $Q_M = span\{\phi_k(\varepsilon): k = 0,1,\cdots, M\}$. The Galerkin approximation of (1) is given by

$$(\upsilon_M'', q)_w + \rho(D^\gamma\, \upsilon_M, q)_w + \sigma(\upsilon_M, q)_w = (g, q)_w; \forall q \in Q_M. \tag{26}$$

Let us denote

$$s_{i,j} = (\phi_i'', \phi_j)_w, y_{i,j} = (D^\gamma\, \phi_i, \phi_j)_w, z_{i,j} = (\phi_i, \phi_j)_w, g_j = (g, \phi_j)_w, \tag{27}$$

then the linear system (26) is equivalent to

$$(\mathbf{S} + \rho\mathbf{Y} + \sigma\mathbf{Z})\mathbf{a} = \mathbf{g}, \tag{28}$$

where,

$$\mathbf{S} = (s_{ij})_{0\le i,j\le M}, \mathbf{Y} = (y_{ij})_{0\le i,j\le M},$$

$$\mathbf{Z} = (z_{ij})_{0\le i,j\le M}, \mathbf{a} = (a_i)_{0\le i\le M}\ and\ \mathbf{g} = (g_i)_{0\le i\le M}.$$

Theorem 1. *The nonzero entries of the matrices* $\boldsymbol{S}$, $\boldsymbol{Y}$, *and* $\boldsymbol{Z}$ *are provided respectively as*

$$s_{ij}=\begin{cases}0, & (i-j)\text{ odd},\\ -8(3+2i)^2, & j=i,\\ -8(3+2i)(3+2j), & (i-j)\text{ even}, i>j,\\ -\dfrac{8(-1)^{2i}(1+i)(2+i)(3+2i)(3+2j)}{(1+j)(2+j)}, & (i-j)\text{ even}, j-1>i,\end{cases}$$

$$y_{ij}=\frac{2(-1)^i(2i+3)(2j+3)}{(j+1)(j+2)}\times$$
$$\left(\frac{\pi\, 4^{\gamma}\gamma\Gamma(-\gamma)}{\Gamma\left(\frac{1}{2}(-i-j-\gamma-1)\right)\Gamma\left(\frac{1}{2}(i-j-\gamma+2)\right)\Gamma\left(\frac{1}{2}(-i+j-\gamma+2)\right)\Gamma\left(\frac{1}{2}(i+j-\gamma+5)\right)}-\right.$$
$$\left.\frac{2}{\Gamma(2-\gamma)}+\frac{2\Gamma(1-\gamma)}{\Gamma(-i-\gamma)\Gamma(i-\gamma+3)}-\frac{2(-1)^j\sin(\pi\gamma)\Gamma(1-\gamma)\Gamma(j+\gamma+1)}{\pi\Gamma(j-\gamma+3)}\right),$$

$$z_{ii}=\frac{4(2i+3)}{(i+1)(i+2)}.$$

Proof. Using relations in (27), we obtain

$$\begin{aligned} s_{ij} &= \int_0^1 \phi_i''(\varepsilon)\phi_j(\varepsilon)w(\varepsilon)d\varepsilon \\ &= \int_0^1 \frac{4(3+2i)(3+2j)\psi_{i+1}^1(\varepsilon)\psi_{j+1}^1(\varepsilon)}{(1+i)(2+j)\varepsilon(\varepsilon-1)}d\varepsilon \\ &= \begin{cases}0, & (i-j)\text{ odd},\\ -8(3+2i)^2, & j=i,\\ -8(3+2i)(3+2j), & (i-j)\text{ even}, i>j,\\ -\frac{8(-1)^{2i}(1+i)(2+i)(3+2i)(3+2j)}{(1+j)(2+j)}, & (i-j)\text{ even}, j-1>i,\end{cases} \end{aligned} \tag{29}$$

where $\psi_{\mu}^{\iota}(\varepsilon)$ is associated Legendre polynomial. We can obtain the rest of the matrices by the same steps. □

Algorithm 1 Creation of an algorithm for our fractional-order Bagley-Torvik method of differential equations

1: Input M.
2: Step 1. Turn the non-homogeneous boundary conditions (2) to Equation (12) using Relation (9).
3: Step 2. Consider that the approximate solution is $u_M(\varepsilon)=\sum_{k=0}^{M} a_k\phi_k(\varepsilon)$.
4: Step 3. Use the matrices **S**, **Y**, **Z**, and **g**.
5: Step 4. Create the residual of the Equation (1).
6: Step 5. Utilize the Galerkin method to get a system in (28).
7: Step 6. Solve the system in (28) by any suitable algebraic method to obtain **a**.
8: Step 7. Set up the approximate solution $u_M(\varepsilon)$.
9: Output $v_M(\varepsilon)$.

6. Discussion of the error and convergence analysis

This section examines the provided method's error and convergence analysis employing several theorems.

Theorem 2. *If* $\omega(\varepsilon)\in C^p[0,1]$, *for some* $p>2$, *then we have the expansion coefficients* a_k *(22),* $\phi_k(\varepsilon)$, $\phi_k''(\varepsilon)$, *and* $D^{\gamma}\phi_k(\varepsilon)$ *satisfy the following estimates:*

$$
\begin{aligned}
&|a_k| \lesssim 2^{-p}(1-p)^{-2k},\\
&|\phi_k(\varepsilon)| \lesssim \frac{8^k}{k!},\\
&|\phi_k''(\varepsilon)| \lesssim 121p^{-1}M^{5-p}(k+1)^3,\\
&|D^\gamma\,\phi_k(\varepsilon)| \lesssim 400p^{-1}M^{4-p}(k+1)^2.
\end{aligned}
\tag{30}
$$

Proof. From the relation (22) and the properties of orthogonality for $\phi_k(\varepsilon)$, we have

$$
\begin{aligned}
|a_k| &= \frac{(k+1)(k+2)}{4(2k+3)}\int_0^1 \phi_k(\varepsilon)\phi_j(\varepsilon)w(\varepsilon)d\varepsilon\\
&= \frac{(k+1)(k+2)}{4(2k+3)}\begin{cases}0, & j\neq k,\\ \frac{4(2k+3)}{(k+1)(k+2)}, & j=k,\end{cases}\\
&\lesssim 2^{-p+3}(2-p)^{-2k}.
\end{aligned}
\tag{31}
$$

In the relation (23), which that gives the next equation

$$
\begin{aligned}
|\phi_k(\varepsilon)| &= \left|\sum_{i=0}^{k+2}\frac{(-1)^{k+2+i}(k+2+i)!}{(k+2-i)!(i!)^2}\varepsilon^i - \sum_{i=0}^{k}\frac{(-1)^{k+i}(k+i)!}{(k-i)!(i!)^2}\varepsilon^i\right|;\, k=0,1,\cdots M\\
&\lesssim \left|4\left(\frac{8^k}{k!}\right)-3\left(\frac{8^k}{k!}\right)\right|\\
&= \frac{8^k}{k!}.
\end{aligned}
\tag{32}
$$

We can prove $\phi_k''(\varepsilon)$ and $D^\gamma\,\phi_k(\varepsilon)$ using the same estimation method in previous relations. □

Theorem 3. *If* $\upsilon(\varepsilon)$ $satisfies$ *the hypothesis of Theorem 2, then the maximum absolute error* (E_M) *is given as next*

$$E_M = \max_{\varepsilon\in[0,1]}|\upsilon(\varepsilon)-\upsilon_M(\varepsilon)|,$$

then we have the following estimate

$$E_M \lesssim 2^{-p+3}(3p-1)^9M^{-1-p}.$$

Proof. The maximum absolute error is specified as

$$\max_{\varepsilon\in[0,1]}|\upsilon(\varepsilon)-\upsilon_M(\varepsilon)| = \max_{\varepsilon\in[0,1]}\left|\sum_{k=M+1}^{\infty} a_k\phi_k(\varepsilon)\right| \tag{33}$$

From **Theorem 2**, we use the relations of a_k and $\phi_k(\varepsilon)$, then we have

$$
\begin{aligned}
E_M &= 2^{-p+3}\sum_{k=M+1}^{\infty}(2-p)^{-2k}\frac{8^k}{k!} = 2^{-p+3}\sum_{k=M+1}^{\infty}\frac{(8(2-p)^{-2})^k}{k!}\\
&\lesssim 2^{-p+3}e^{8(2-p)^{-2}}\left(1-\frac{\Gamma(8(2-p)^{-2},M+1)}{\Gamma(M+1)}\right)\\
&\lesssim 2^{-p+3}(3p-1)^9M^{-1-p},
\end{aligned}
\tag{34}
$$

where $\Gamma(\eta,\Xi)$ represents the upper incomplete gamma function. □

Theorem 4. *If* $p>5$ *and* $\upsilon(\varepsilon)$ *supports the hypotheses of Theorem*

$$R_M = \max_{\varepsilon\in[0,1]}|\upsilon_M''(\varepsilon)+\rho\,D^\gamma\upsilon_M(\varepsilon)+\sigma\upsilon_M(\varepsilon)-g(\varepsilon)|,$$

then we have the following estimate: $R_M \lesssim 2^{-p}p^{-1}M^{5-p}$.

Proof. We have the following form of R_M, which is

$$R_M = \max_{\varepsilon\in[0,1]}\left|\sum_{k=0}^{M}a_k\phi_k''(\varepsilon)+\rho\sum_{k=0}^{M}a_kD^\gamma\phi_k(\varepsilon)+\sigma\sum_{k=0}^{M}a_k\phi_k(\varepsilon)-g(\varepsilon)\right| \tag{35}$$

Now, we use the hypotheses of $|a_k|$, $|\phi_k''(\varepsilon)|$, $|D^\gamma\,\phi_k(\varepsilon)|$, and $|\phi_k(\varepsilon)|$, which there exist in Theorem 2, then we have the next relation

$$\begin{aligned} R_M \quad &\lesssim \left| 2^{-p+3}(2-p)^{-2k}\left(121p^{-1}M^{5-p}(i+1)^3 + 400\rho p^{-1}M^{4-p}(i+1)^2 + \sigma\frac{8^k}{k!}\right)\right| \\ &\lesssim 2^{-p+3}(3p-1)^9\, M^{-3-p}. \square \end{aligned} \tag{36}$$

7. Some practice problems for the fractional Bagley-Torvik equation

In this part, the shifted Legendre polynomials on second-order fractional differential problems are examined to determine the absolute error. Mathematica's version 11 software solved each of the following examples.

Example 1. *We study the Bagley-Torvik differential equation that follows* [26–28]:

$$D^{\frac{3}{2}}\upsilon(\varepsilon) + \upsilon(\varepsilon) = \varepsilon^2 - \varepsilon + \frac{2\varepsilon^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}, 0 \leqslant \varepsilon \leqslant 1, \tag{37}$$

under the constraint of homogeneous boundary conditions:

$$\upsilon(0) = \upsilon(1) = 0, \tag{38}$$

the analytical solution to the Bagley-Torvik differential Equation (37) using the boundary conditions (38) is provided via

$$\upsilon(\varepsilon) = \varepsilon^2 - \varepsilon. \tag{39}$$

For M = 1, the operational matrix $\mathbf{Y} = (D^{\frac{3}{2}}\phi_i, \phi_j)_w$ is given by

$$\mathbf{Y} = \begin{pmatrix} -54.16220002 & -18.05406667 \\ 54.16220002 & -64.47880955 \end{pmatrix}, \tag{40}$$

the residual of (37) represents as

$$R_j = \left(\sum_{i=0}^{1} a_i\left(y_{i,j} + \frac{4(3+2j)}{(1+j)(2+j)}\delta_{i,j}\right)\right) - g_j, j = 0, 1, \tag{41}$$

where the residual produces a system consisting of two linear equations are provided by

$$\begin{aligned} &(-16+\sqrt{\pi})(6a_0 - 1) + 96a_1 = 0, \\ &336a_0 + 5(240 - 7\sqrt{\pi})a_1 = 56, \end{aligned} \tag{42}$$

then the solution of the linear system is $a_0 = 0.166667$ and $a_1 = 0$, so the approximate solution is given by

$$\begin{aligned} \upsilon_1(\varepsilon) \quad &= a_0\phi_0(\varepsilon) + a_1\phi_1(\varepsilon) = 0.166667(1 - 6\varepsilon + 6\varepsilon^2 - 1) \\ &= 1.000002(\varepsilon^2 - \varepsilon) \approx \varepsilon^2 - \varepsilon, \end{aligned} \tag{43}$$

where that is the analytical solution.

Example 2. *Let us examine the fractional Bagley-Torvik differential equation* [14,29]

$$\upsilon''(\varepsilon) + D^{\gamma}\,\upsilon(\varepsilon) + \upsilon(\varepsilon) = \varepsilon^2 + 2 + 4\sqrt{\frac{\varepsilon}{\pi}}, 0 \leqslant \varepsilon \leqslant 1, 1 < \gamma < 2, \tag{44}$$

given that the following homogeneous boundary conditions apply:

$$\upsilon(0) = 0, \upsilon(1) = 1, \tag{45}$$

the analytical solution when $\gamma = 1.5$ is

$$\upsilon(\varepsilon) = \varepsilon^2. \tag{46}$$

First, we change the boundary conditions from non-homogeneous to homogeneous by relation (9), then we have

$$u''(\varepsilon)+D^{1.5}u(\varepsilon)+u(\varepsilon)=\varepsilon^2+2+4\sqrt{\frac{\varepsilon}{\pi}}-\varepsilon, \tag{47}$$

by the homogeneous boundary conditions

$$u(0)=0,u(1)=0, and\ u(\varepsilon)=\varepsilon^2-\varepsilon. \tag{48}$$

For $M=4$, we apply our method to differential Equation (44) with their boundary conditions (45), respectively, the operational matrices for $D^{1.5}u(\varepsilon)$, $u''(\varepsilon)$, $u(\varepsilon)$, and $f(\varepsilon)$ are given by

$$\mathbf{Y}=\begin{pmatrix} -\frac{96}{\sqrt{\pi}} & -\frac{32}{\sqrt{\pi}} & -\frac{32}{\sqrt{\pi}} & -\frac{96}{5\sqrt{\pi}} & -\frac{96}{5\sqrt{\pi}} \\ -\frac{96}{\sqrt{\pi}} & -\frac{800}{7\sqrt{\pi}} & -\frac{224}{9\sqrt{\pi}} & -\frac{3744}{77\sqrt{\pi}} & -\frac{2464}{117\sqrt{\pi}} \\ -\frac{192}{\sqrt{\pi}} & -\frac{448}{9\sqrt{\pi}} & -\frac{2240}{11\sqrt{\pi}} & -\frac{4032}{65\sqrt{\pi}} & -\frac{320}{3\sqrt{\pi}} \\ -\frac{192}{\sqrt{\pi}} & -\frac{12480}{77\sqrt{\pi}} & -\frac{1344}{13\sqrt{\pi}} & -\frac{88128}{385\sqrt{\pi}} & -\frac{10560}{221\sqrt{\pi}} \\ -\frac{288}{\sqrt{\pi}} & -\frac{12320}{117\sqrt{\pi}} & -\frac{800}{3\sqrt{\pi}} & -\frac{15840}{221\sqrt{\pi}} & -\frac{91168}{285\sqrt{\pi}} \end{pmatrix}, \tag{49}$$

$$\mathbf{S}=\begin{pmatrix} -72 & 0 & -28 & 0 & -\frac{88}{5} \\ 0 & -200 & 0 & -108 & 0 \\ -168 & 0 & -392 & 0 & -\frac{1232}{5} \\ 0 & -360 & 0 & -648 & 0 \\ -264 & 0 & -616 & 0 & -968 \end{pmatrix}, \tag{50}$$

$$\mathbf{Z}=\begin{pmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & \frac{10}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{9}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{22}{15} \end{pmatrix}, \tag{51}$$

$$\mathbf{f}=\begin{pmatrix} -11-\frac{16}{\sqrt{\pi}} \\ -\frac{16}{3\sqrt{\pi}} \\ \frac{2}{3}(-7-\frac{8}{\sqrt{\pi}}) \\ -\frac{16}{5\sqrt{\pi}} \\ -\frac{44}{15}-\frac{16}{5\sqrt{\pi}} \end{pmatrix}. \tag{52}$$

Then the residual of Equation (47)

$$R_j=\left(\sum_{i=0}^{j} a_i(\mathbf{S}+\mathbf{Y}+\mathbf{Z})\right)-\mathbf{f}^T, j=0,1,2,3,4. \tag{53}$$

When we follow the Galerkin method, we obtain a linear system of five equations in five unknowns are introduced as the following

$$\begin{aligned} &-16+212.9819542a_0-96a_1+489.772247a_2-192a_3+755.9278166a_4=0,\\ &-24024+144144a_0+2084990.28a_1-224224a_2+3604326.614a_3-474320a_4=0,\\ &-17509.35783+105056.147a_0+32032a_1+1150967.379a_2-133056a_3+1748387.233a_4=0,\\ &-272272+1633632a_0+204245187.48a_1+103488a_2+2292729.73a_3-6098400a_4=0,\\ &-793535.2255+4761211.353a_0+1989680a_1+51339006.94a_2+4514400a_3+192074972.8a_4=0. \end{aligned} \tag{54}$$

The solution of the previous linear system is $a_0=0.166667$, $a_1=0$, $a_2=0$, $a_3=0$, and $a_4=0$. Therefore, the approximate solution is

$$\begin{aligned} u_4(\varepsilon) &= a_0\phi_0(\varepsilon) + a_1\phi_1(\varepsilon) + a_2\phi_2(\varepsilon) + a_3\phi_3(\varepsilon) + a_4\phi_4(\varepsilon) \\ &= 0.166667(1 - 6\varepsilon + 6\varepsilon^2 - 1) = 1.000002(\varepsilon^2 - \varepsilon) \approx \varepsilon^2 - \varepsilon, \end{aligned} \quad (55)$$

and that is the analytical solution.

Example 3. *Let's look at the Bagley-Torvik problem [30]*

$$v''(\varepsilon) + \frac{8}{17}D^\gamma v(\varepsilon) + \frac{13}{51}v(\varepsilon) = f(\varepsilon), 0 \leqslant \varepsilon \leqslant 1, 1 < \gamma < 2, \quad (56)$$

depending on the homogeneous boundary conditions:

$$v(0) = v(1) = 0, \quad (57)$$

where $f(\varepsilon)$ is selected so that the analytical solution is supplied by

$$v(\varepsilon) = \varepsilon^5 - \frac{29}{10}\varepsilon^4 + \frac{76}{25}\varepsilon^3 - \frac{339}{250}\varepsilon^2 + \frac{27}{125}\varepsilon. \quad (58)$$

We employ our method to Equation (56) with the homogeneous boundary conditions (57) for $M = 3$ and $M = 4$, therefore we contrast the absolute error in the method we use with others as shown in **Table 1**. **Table 2** shows the running times of our method to various values of M. **Figure 1** depicts the shifted Legendre expansion behavior, demonstrating the absolute error at $M = 3$. In **Figure 2**, we contrast the analytical with approximate solutions with $M = 3, 4, \dots, 8$. Which the approximate solution at $M = 3$ is quite near to the analytical solution.

Table 1. Our method's absolute error when using various M values for Example 2.

ε	M = 3	M = 4	M = 8	M = 256
0.1	0	0	3.597×10^{-4}	3.899×10^{-6}
0.2	8.882×10^{-16}	5.329×10^{-15}	1.583×10^{-3}	4.171×10^{-6}
0.3	0	7.105×10^{-14}	1.787×10^{-3}	3.943×10^{-6}
0.4	0	5.684×10^{-13}	1.634×10^{-3}	3.373×10^{-6}
0.5	2.274×10^{-13}	2.956×10^{-12}	1.158×10^{-3}	2.609×10^{-6}
0.6	1.819×10^{-12}	1.000×10^{-11}	5.836×10^{-4}	1.788×10^{-6}
0.7	3.638×10^{-12}	2.910×10^{-11}	1.271×10^{-4}	1.041×10^{-6}
0.8	7.276×10^{-12}	6.912×10^{-11}	1.197×10^{-4}	4.920×10^{-7}
0.9	1.455×10^{-11}	1.455×10^{-10}	5.540×10^{-4}	2.611×10^{-7}

Table 2. Running times of our method for Example 2 in seconds.

M	3	4	5	6	7	8
CPU	12.891	14.36	22.782	34.954	82.906	90.938

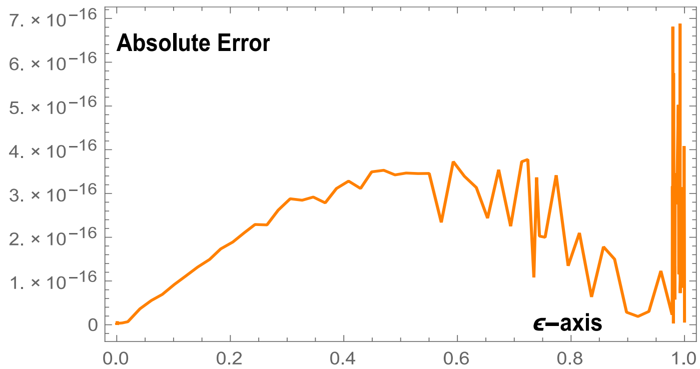

Figure 1. The absolute error with $M = 3$ for Example 3.

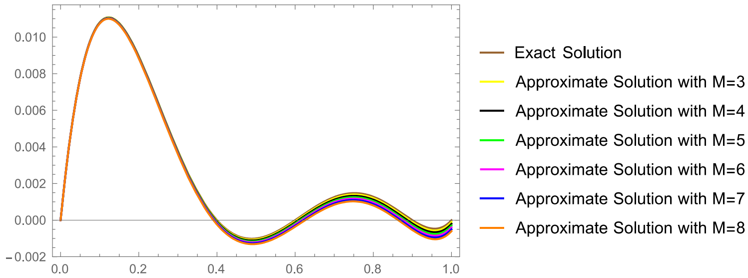

Figure 2. Comparing the approximate and analytical solutions for Example 3 for different values of M.

8. Conclusions

Finding the exact answers to fractional differential equations is typically challenging. Thus, approximating solution methods is required. In our study, we attempted the Galerkin method for solving the fractional Bagley-Torvik problems with shifted Legendre polynomials. Our method can reach the exact solution, in which the absolute error equals zero. The mentioned method converted linear fractional Bagley-Torvik problems into an algebraic system that relied on the Galerkin method. It is easy to solve them by any algebraic method. The method's efficiency in producing results has been demonstrated using instances with exact solutions. The approximation of fractional Bagley-Torvik equations solved by this method is highly practical and effective, as shown by the numerical results and figures. The research used a PC with those specifications: Processor: Intel(R) Core(TM) i7-4790 CPU @ 3.60GHz; installed memory: 16.0 GB.

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