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Miraculous hypercomplex numbers

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ABSTRACT

The development of the number system has been a long and difficult process, and many landmark concepts and theorems have been put forward. By briefly reviewing the development of hypercomplex systems, the rules for constructing the unit elements are discussed. As a vector space defining multiplication, division, and norm of vectors, hypercomplex numbers synthesize the advantages of mathematical tools such as algebra, geometry, and analysis, faithfully describe the intrinsic properties of space-time and physical systems, and provide a unified language and a powerful tool for basic theories and engineering technology. In the application of hypercomplex numbers, the group-like properties of the basis vectors are the most important, and the zero factor has little influence on the algebraic operation. The multiplication table of the basis vectors fully describes the intrinsic properties of the hypercomplex system, and the matrix **A** constructed from the multiplication table satisfies the structure equation $\mathbf{A}^2 = n\mathbf{A}$ and thus obtains a set of faithful matrix representations of the basic elements. This paper also uses typical examples to show the simple and clear concepts and wide application of hypercomplex numbers. Therefore, hypercomplex numbers are worth learning in basic education and applying in scientific research and engineering technology.

Keywords: Clifford algebra; Grassmann algebra; hypercomplex number; structure equation; consistent equation

1. Sketching the history of number systems

The understanding of numbers and number systems has gone through a long and difficult history, and the introductions of some key concepts are milestones in this process. In 500 BC, the Pythagoras school proposed the universal belief that "all is a number", believing that all phenomena in the universe could be attributed to integers or rational numbers of the ratio of integers. However, Hippasus, a member of the school, found that the diagonal length and side length of the square were irreducible, revealing the existence of irrational numbers[1]. In "JiuZhang SuanShu (Nine Chapters of Arithmetic)" in the Han Dynasty of China, the concepts of negative numbers and computing rules have already been described. In 665, Brahmagupta in India allowed the existence of negative number solutions when solving the quadratic equation, but Cardan, Vieta, and Pascal all regarded this as an absurd number. R. Descartes thought that the negative number as the root of the equation should be a false one. Although Wallis accepted the concept of the negative number, he felt that the negative number should be larger than infinity[1]. The completion of the real number theory went through a long time until the 19th century. Only after R. Dedekind, G. Cantor, and K. Weierstrass developed the rigorous theory of irrational numbers was real number theory finally established[2].

In 1545, in order to express the roots of the three- and four-order equations, Geronimo Cardano introduced

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the square root of the negative number. Since there was no intuitive explanation for this number at the time, he calls it "sophistic" and concludes that it is "as subtle as it is useless". In 1637, Descartes called it an imaginary number, and in 1673, John Wallis gave a geometric interpretation of complex numbers. In 1777, Euler introduced the notation $\mathrm{i} = \sqrt{-1}$. At the beginning, the imaginary numbers were rejected by many mathematicians. Descartes believed that the appearance of complex roots meant that the equation was unsolvable; G.W. Leibniz said that they were a sort of amphibian, halfway between existence and nonexistence. This is somewhat similar to Schrödinger's cat in quantum mechanics. In 1831, K.F. Gauss proposed the geometrical representation of complex numbers, that is, the points in a complex plane, which is widely used in solving practical problems. This new number system was widely recognized and accepted. From the perspective of algebra, the rational, real, and complex numbers are all number fields with operations of addition, subtraction, multiplication, and division and do not have a zero factor, and the computations satisfy the associativity, commutativity, and distributive law, so their algebraic properties can be said to be perfect.

By the 19th century, people had a clear understanding of numerical computations and algebraic operations and began to study the legitimacy of symbolic operations. The "Report on the recent progress and present state of certain branches of analysis", published by Peacock in 1834, clarified certain permanence principles of symbolic algebra, which paved the way for the later development of abstract algebra, especially Boolean algebra. In 1841, De Morgan published "On the Foundation of Algebra", introducing symbolic algebra to explain the operation of a specific algebra, considering $(+,-,\times,/)$ and zero, and concepts such as commutativity, associativity, and distributivity, studying the correct axiomatic treatment of equality.

How to extend the superiority of complex numbers in the plane to 3-dimensional space was a difficult problem in front of people at that time, and many famous mathematicians were looking for "3-ary numbers". The Irish mathematician Hamilton[3] also joined the ranks in the search for 3-ary numbers due to the actual needs of physics. After 15 years of trying and thinking, on October 16, 1843, he finally introduced ordered arrays of four real numbers, abandoned the commutativity of multiplication in the new number system, and defined the first hypercomplex number—quaternion[3]

$$q = a + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, \qquad \boldsymbol{i}^2 + \boldsymbol{j}^2 + \boldsymbol{k}^2 = \boldsymbol{ijk} = -1.$$

With real numbers, complex numbers, and quaternions, a natural idea is to similarly expand logarithmically, abandoning algebraic properties as little as possible. Soon after the discovery of quaternions, the concepts of double complex systems, biquaternions, complex quaternions, and octonions were discovered.

In 1848, Sylvester first used the term matrix to represent an array of numbers. In 1855, Cayley studied the matrix of the linear transformation and defined the matrix multiplication. In 1861, Weierstrass showed that, keeping all algebraic properties, the complex number is the only finite-dimensional extension of the real number. In 1878, Frobenius proved an important theorem: $(\mathbb{R}, \mathbb{C}, \mathrm{H})$ is the only finite-dimensional associative division algebra over $\mathbb{R}$ without zero factor[4]. A generalized Frobenius theorem, proved by Hurwitz[5], Milnor[6], pointed out that if the associativity of multiplication is abandoned again, the algebra with modular product law and unit element is only an octonion or Cayley number.

To improve the limitations of quaternion algebra applied in physics, in 1944, Gibbs published the "Vector Analysis"[7], while Heaviside published "Electromagnetic Induction and its Propagation", re-expressed Maxwell's theory of electrodynamics, developed the modern vector calculus, and promoted the application of vector algebra in physics[8]. A further extension of the number system is general associative algebras. In 1844, Hermann Grassmann began to study exterior algebra, dealing with the geometric problem in the *n*-dimensional vector space[9]. He defines the inner and exterior products of vectors (**a, b**) by

$$\boldsymbol{a}\cdot\boldsymbol{b}=\sum_{k=1}^{n}a_k\,b_k,\qquad \boldsymbol{a}\wedge\boldsymbol{b}=\sum_{j,k=1}^{n}a_j\,b_k\boldsymbol{e}_j\wedge\boldsymbol{e}_k=-\boldsymbol{b}\wedge\boldsymbol{a}. \tag{1}$$

Their geometric meanings are as follows: $|\boldsymbol{a}|=\sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$ indicates the length of the vector **a**, while $\boldsymbol{a}\wedge\boldsymbol{b}$ represents the area of a parallelogram composed of **a** and **b** as the edges. The Grassmann algebra $\bigoplus_{k=0}^{n}\Lambda^k(V)$ is a 2^n-dimensional algebra, with each term having a clear geometric meaning.

In 1878, Clifford proposed a modification to Grassmann algebra, which combines quaternion and Grassmann algebra and is now known as Clifford algebra[10]. Clifford algebra is closely related to *n*-dimensional geometry, so he call it geometric algebra himself[11]. He defines the algebra by the following Clifford relation

$$\boldsymbol{e}_a\boldsymbol{e}_b+\boldsymbol{e}_a\boldsymbol{e}_b=2\eta_{ab}\boldsymbol{I},\qquad \boldsymbol{e}^a\boldsymbol{e}^b+\boldsymbol{e}^a\boldsymbol{e}^b=2\eta^{ab}\boldsymbol{I}, \tag{2}$$

where $(\eta_{ab})=(\eta^{ab})=\mathrm{diag}(\boldsymbol{I}_p,-\boldsymbol{I}_q)$ is the metric of n-dimensional Minkowski spacetime, $\boldsymbol{e}_a\boldsymbol{e}_b$ is Clifford product. For any vectors $(\mathbf{a},\mathbf{b})$, by Equation (2) we have

$$\mathbf{ab}=\frac{1}{2}(\mathbf{ab}+\mathbf{ba})+\frac{1}{2}(\mathbf{ab}-\mathbf{ba})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}. \tag{3}$$

Clifford algebra is also a 2^n-dimensional associative algebra, equivalent to the Grassmann algebra in the sense of linear algebra. Unlike Grassmann algebra, Clifford algebra is isomorphic to some special matrix algebra, and the geometric product is directly corresponding to the matrix product.

The classification of associative algebra began in 1872 with the work of Peirce[12] and his son, using the nilpotent $A^n = 0$ and idempotent $A^2 = A$ to classify the algebra and construct many specific associative algebras through the unit element multiplication table. If i is an idempotent $\mathrm{i}^2 = \mathrm{i}$, every element A of the algebra can be written as the right Peirce decomposition

$$A=B+C,\quad B=\mathrm{i}A,\quad C=A-\mathrm{i}A,$$

thus, we have $\mathrm{i}B = B$ and $\mathrm{i}C = 0$. A complete list of algebras with unity of dimensions up to 4 over the fields $\mathbb{R}$ and $\mathbb{C}$ was presented by the German geometer Study[13]. He added still another variant to the collection of complex products. The "dual" numbers arose from the convention that $\mathrm{i}^2 = 0$[14,15]. Then we obtain a multiplication rule different from the ordinary complex numbers as

$$(a+b\mathrm{i})(x+y\mathrm{i})=ax+(ay+bx)\mathrm{i}. \tag{4}$$

In 1854, Cayley introduced what we today call the group algebra of a finite group G. The basis elements of this algebra are just the group elements $\{g_k; k=1,\cdots,n\}$, with the multiplication rule $g_jg_k=g_i$. Every representation of the group G by linear transformations can be extended to a representation of the group algebra. Conversely, every representation of the group algebra yields a representation of the group. Therefore, the study of the structure of group algebra is of primary importance in the theory of group representations[16].

The first to investigate the structure of the group algebra of a finite group was Theodor Molien. In 1893–1897, Molien proved several fundamental theorems concerning the structure of algebras over the complex fie1d $\mathbb{C}$, he applied his general theory to the group algebra of a finite group[17,18]. Molien[18] and Frobenius[19] proved independently that the finite group algebra is a direct sum of full matrix algebras, and from this, they concluded that every representation of the algebra is completely reducible, and that every irreducible representation is contained in the regular representation[19]. Several results obtained by Molien were rediscovered by Cartan in 1898[20], Cartan's theory culminates in two theorems. The first theorem says, in modern terminology: Every simple algebra over $\mathbb{C}$ is a full matrix algebra. Cartan defines a semi-simple algebra as a direct sum of simple algebras. His second theorem says: Every algebra over $\mathbb{C}$ is a direct sum of a simple or semi-simple subalgebra and a nilpotent invariant subalgebra[21].

The first to develop a general theory of algebras over an arbitrary field was Wedderburn. He defines the invariant subcomplex B of A, we call it a two-sided ideal by $AB \subseteq B$ and $BA \subseteq B$. Every such ideal defines a residue-class algebra *A/B*. He proves that all simple associative algebras over a field F are precisely the full matrix algebras with elements from an associative division algebra over F[22]. Wedderburn's structural theorem says:

Any algebra is the sum of its radical *N* and a semi-simple algebra.

A semi-simple algebra can be uniquely expressed as a direct sum of simple algebras.

A simple algebra is a full matrix algebra over a division algebra.

The Cayley-Dickson construction generates a new algebraic system sequence of complex, quaternion, and octonion starting from the real numbers[23,24]. Each algebra has the concepts of norm and conjugation, and the dimension of each algebraic system in the sequence is 2 times that of its predecessor. For example, in the construction of quaternions, the ordered pair of complex numbers (*a, b*), addition is defined as the addition of the corresponding components, while multiplication is defined as

$$(a,b)(c,d) = (ac - b\bar{d}, ad + b\bar{c}).$$

The conjugate of (*a, b*) is defined as $(a,b)^+ = (\bar{a}, -b)$, then we have

$$(a,b)^+(a,b) = (\bar{a}, -b)(a,b) = (\bar{a}a + \bar{b}b, b\bar{a} - b\bar{a}) = (|a|^2 + |b|^2, 0).$$

Quaternions consist of two independent complex numbers which constitute a 4-dimensional vector space over $\mathbb{R}$. The multiplication of the quaternions is non-commutative. Starting with the quaternions, and repeating the above steps, we can construct octonions. Since an octonion consists of two independent quaternions, they constitute an 8-dimensional vector space over $\mathbb{R}$. The multiplication of octonions is non-associative, so it cannot be regarded as a "number system" in the normal sense. The subsequent algebras remain a power-associative property, but lose the property as an alternative algebra, and thus is no longer a synthetic algebra. Cayley-Dickson construction can continue, and each step produces a power-associative algebra with twice the dimension of the previous algebra.

2. Rules to construct basis elements

From the classical pieces of literature, we cannot find a clear definition of hypercomplex numbers. Hypercomplex numbers usually refer to the finite-dimensional associative algebra and the non-associative algebras such as Cayley-Dickson construction, which are somewhat different from the ordinary number systems. For example, many associative algebras do not have the definitions of norm and division, and the Cayley-Dickson construction larger than 8 dimensions can hardly be treated as a number system. The division of a number system is closely related to the solvability of the equation. For example, for the linear hypercomplex equation, we have

$$\boldsymbol{ax} = \boldsymbol{b}, \qquad (||\boldsymbol{a}|| \neq \boldsymbol{0}) \Leftrightarrow \boldsymbol{x} = \boldsymbol{a}^{-1}\boldsymbol{b}.$$

For the nonlinear hypercomplex equation $f(\mathbf{x}) = 0$, under the appropriate conditions the solution can be solved by the following hypercomplex Newton iteration method[25]

$$\boldsymbol{x_{m+1}} = \boldsymbol{x_m} - \boldsymbol{f}(\boldsymbol{x_m})(\boldsymbol{f}'(\boldsymbol{x_m}))^{-\boldsymbol{1}}.$$

Many complicated systems in Nature are high-dimensional and nonlinear, and it is difficult to describe the intrinsic structure of the system only by vector space. In addition, the parameters of nonlinear systems are rarely globally valid, but they have only continuous dependence and solvability in a certain domain, which is very different from a linear space. These two features should be reflected in the definition of hypercomplex numbers. Specifically, like ordinary numbers, hypercomplex numbers should meet the associativity and distributivity, including unit element **I** and appropriate definitions of norm and reciprocal. Only in this way can they be easily used to solve practical problems, such as solving the root of algebraic equations, and analyzing the hydrodynamics and gauge field equations[26].

Thus, as the standard basis elements of hypercomplex numbers over field F, $\{\boldsymbol{e}_k\}$ should satisfy the following group-like properties as the necessary conditions for effective computation:

Existing unit element: $\boldsymbol{e}_0 = \boldsymbol{I}$, such that $\boldsymbol{I}\boldsymbol{e}_k = \boldsymbol{e}_k\boldsymbol{I} = \boldsymbol{e}_k$.

Associativity: $(\boldsymbol{e}_j\boldsymbol{e}_k)\boldsymbol{e}_m = \boldsymbol{e}_j(\boldsymbol{e}_k\boldsymbol{e}_m)$.

Closure for multiplication

$$\boldsymbol{e}_j\boldsymbol{e}_k = f_{jk}\boldsymbol{e}_m, \qquad |f_{jk}| = 1, \qquad f_{jk} \in \mathbf{F}. \tag{5}$$

Existing generalized inverse elements: $\boldsymbol{e}_k^{-1} = e^{i\theta_k}\boldsymbol{e}_j$, such that

$$\boldsymbol{e}_k\boldsymbol{e}_k^{-1} = \boldsymbol{e}_k^{-1}\boldsymbol{e}_k = \boldsymbol{e}_0. \tag{6}$$

Unitary norm: $||\mathbf{e}_k|| = 1$.

Obviously, the hypercomplex number is first a vector space over F. For a given set of basis vectors $\{\boldsymbol{e}_0 = \boldsymbol{I}, \boldsymbol{e}_k; k = 1, \cdots, n-1\}$, we have hypercomplex numbers

$$\mathbf{x} = \mathbf{x}^a\mathbf{e}_a, \qquad x^a \in \mathrm{F}.$$

Hereafter we adopt the Einstein summation convention, if not specified, the upper and lower double marks indicate the sum of all indicators.

If the hypercomplex has an *m*-th order matrix representation, then the norm of **x** can be defined as $||\boldsymbol{x}|| = \sqrt[m]{|\det(\boldsymbol{x})|}$. This Calvet's norm ||**x**|| is an invariant scalar under the transformation of rotation, reflection, and translation[27]. In fact, for any given unitary matrix, the similarity transformation transforms one set of orthogonal bases to another set of orthogonal bases. By the multiplication rule of the matrix determinant, we have ||**x'**|| = ||**x**|| and the modulus product law $||\boldsymbol{x}\boldsymbol{y}|| = ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||$. This norm is the same as the usual moduli for ordinary numbers of real, complex, and quaternions. Easy to prove that the set of {||**x**|| = 0} is a low-dimensional closed set. The zero norm set {det(**x**) = 0} is some analytic hypersurfaces similar to light cones, which has little influence on the algebraic operation, far from the serious problem of abandoning the associativity[25].

For Pauli matrices

$$\sigma_a \in \left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right\}, \tag{7}$$

their multiplication rule is as

$$\sigma_a^2 = \boldsymbol{I}, \qquad \sigma_1\sigma_2 = -\sigma_2\sigma_1 = \mathrm{i}\sigma_3, \qquad \sigma_a\sigma_b = \epsilon_{abc}\mathrm{i}\sigma_c.$$

The coefficients f_{jk} contain the complex units i, so $\mathbf{x} = x^a\sigma_a$ constitutes the quaternions over $\mathbb{C}$. If taking all the following matrices as basic elements

$$\boldsymbol{e}_a = (\boldsymbol{I}, \sigma_j, \mathrm{i}\sigma_k, \mathrm{i}\boldsymbol{I}), \tag{8}$$

then $f_{jk} = \pm 1$ and

$$\boldsymbol{x} = s\boldsymbol{I} + E^a\sigma_a + B^b\mathrm{i}\sigma_b + p\mathrm{i}\boldsymbol{I}$$

constitutes a class of associative octonions over $\mathbb{R}$. We have

$$\det(\boldsymbol{x}) = s^2 - p^2 - \vec{E}^2 + \vec{B}^2 + 2\mathrm{i}(sp - \vec{E}\cdot\vec{B}),$$

$$||\boldsymbol{x}|| \equiv \sqrt{|\det(\boldsymbol{x})|} = \sqrt[4]{(s^2 - p^2 - \vec{E}^2 + \vec{B}^2)^2 + 4(sp - \vec{E}\cdot\vec{B})^2}.$$

This hypercomplex system is isomorphic to Clifford algebra $C\ell(\mathbb{R}^{3,0})$, and the imaginary unit i appearing in the determinant has no effect on neither the hypercomplex calculation nor the norm calculation[28,29]. In the next section, we examine some applications of this kind of hypercomplex number in physics.

If taking $\{\boldsymbol{I}_2, \boldsymbol{i} = \mathrm{i}\sigma_1, \boldsymbol{j} = -\mathrm{i}\sigma_2, \boldsymbol{k} = \mathrm{i}\sigma_3\}$ as bases, then we have multiplication rules as

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{I}.$$

So, we obtain the quaternions over real field $\mathbb{R}$, which is isomorphic to the Clifford algebra $C\ell(\mathbb{R}^{0,2})$. Due to the group-like properties of the multiplication of basis matrices, the number field of the hypercomplex coordinates and the number field of the basis matrices is sometimes independent of each other. If the basis multiplication of a hypercomplex number is $\boldsymbol{e}_j\boldsymbol{e}_k = f_{jk}\boldsymbol{e}_m$, then in the case of $f_{jk} = \pm 1$ the coordinates x^a can be defined on any number field F, but in the case $f_{jk} = e^{\mathrm{i}\theta} \notin \mathbb{R}$ the coordinates $x^a \in \mathbb{C}$.

The properties of the hypercomplex numbers are completely determined by the multiplication table of the basis vectors. Converting the multiplication table into multiplication matrix we get

$$\boldsymbol{M} \equiv \boldsymbol{e}^T\boldsymbol{e}, \qquad \boldsymbol{e} = (\boldsymbol{e}_0, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n-1}), \tag{9}$$

then **M** fully describes the algebraic properties of the basic elements. For the normal bases that satisfy the above group-like properties, we have the following basic conclusions[26].

Theorem 1. *If the bases* $\{\mathbf{e}_k\}$ *satisfy the above group-like properties, and* **M** *is the multiplication matrix of the bases, denote*

$$\boldsymbol{C}^m = \frac{\partial \boldsymbol{M}}{\partial \boldsymbol{e}_m}, \qquad \boldsymbol{E}^m = \boldsymbol{C}^m(\boldsymbol{C}^0)^{-1}, \qquad \boldsymbol{A} = \boldsymbol{M}(\boldsymbol{C}^0)^{-1} = \boldsymbol{E}^m\boldsymbol{e}_m, \tag{10}$$

then we have structure equation $\boldsymbol{A}^2 = n\boldsymbol{A}$, *and* $\boldsymbol{E}_m \equiv \overline{\boldsymbol{E}}^m \leftrightarrow \boldsymbol{e}_m$ *is an isomorphic map.* $\{\mathbf{E}_k\}$ *is a faithful matrix representation of* $\{\mathbf{e}_k\}$ *satisfying* $|\det(\mathbf{E}_k)| = 1$.

By the above theorem, for any given multiplication table of elements, we can establish the multiplication matrix **M** and $\boldsymbol{A} = \boldsymbol{M}(\boldsymbol{C}^0)^{-1}$. If $\mathbf{A}^2 = n\mathbf{A}$, then the canonical matrix representation $\{\mathbf{E}_k\}$ can be defined and then we can establish a hypercomplex number system by using $\{\mathbf{E}_k\}$. By Equation (2) we find $\boldsymbol{C}^0 = (\boldsymbol{C}^0)^T$. For $\boldsymbol{B} = (\boldsymbol{C}^0)^{-1}\boldsymbol{A}\boldsymbol{C}^0 = (\boldsymbol{C}^0)^{-1}\boldsymbol{M}$, we have conclusions similar to Theorem 1. The condition $\boldsymbol{e}_j\boldsymbol{e}_k = f_{jk}\boldsymbol{e}_m$ guarantees that the inverse element $\mathbf{e}_m^{-1}$ is also a monomial, otherwise it will be too complicated to form a number system.

The calculations show that the multiplication matrix **M** has some interesting properties. For example, $||\det(\mathbf{M})|| = 0$ may be a necessary and sufficient condition for number system having zero factor. Assume the matrices $\{\mathbf{E}_k\}$ are the canonical representation of the normal bases $\{\mathbf{e}_k\}$, by $\mathbf{A} = \mathbf{M}(\mathbf{C}^0)^{-1} = \mathbf{E}^m\mathbf{e}_m$, we have $||\det(\mathbf{M})|| = ||\det(\mathbf{A})||$. For complex numbers we have $||\det(\mathbf{M})|| = 2$, and for quaternions we have $||\det(\mathbf{M})|| = 4$. However, for the hyperbolic number of $\mathbf{e}_1^2 = \mathbf{e}_0$ we have

$$||\det(\mathbf{M})|| = ||\det\begin{pmatrix} \mathbf{e_0} & \mathbf{e_1} \\ \mathbf{e_1} & \mathbf{e_0} \end{pmatrix}|| = \mathbf{0}.$$

In the case of any basis without inverse element, e.g., Equation (4), the multiplication matrix is singular.

The simplest hypercomplex numbers are the following commutative cyclic numbers

$$\boldsymbol{A_n} = \sum_{\boldsymbol{k=0}}^{\boldsymbol{n-1}} \boldsymbol{a_k}\, \boldsymbol{e_k}, \qquad (\boldsymbol{a_0}, \boldsymbol{a_1}, \cdots, \boldsymbol{a_{n-1}}) \in \boldsymbol{F^n}, \tag{11}$$

in which

$$\boldsymbol{e}_0 = \boldsymbol{I}_n, \qquad \boldsymbol{e}_m = \begin{pmatrix} 0 & \boldsymbol{I}_{n-m} \\ \boldsymbol{I}_m & 0 \end{pmatrix}, \qquad 1 \le m \le n-1.$$

$\{\boldsymbol{e}_m; m = 0, 1, \dots, n-1\}$ is a matrix representation of *n*-element cyclic group. The roots of an *n*-th order algebraic equation can be expressed by Equation (11), according to Gu[25]

Theorem 2. *Let* $w = \exp(2\pi i/n)$, *for n-ary number* (11) *over* $\mathbb{R}$ *or* $\mathbb{C}$, *denoting*

$$R_k = a_0 + a_1 w^k + a_2 w^{2k} + \cdots + a_{n-1} w^{(n-1)k},$$

where $(k = 0, 1, 2, \dots, n-1)$, *then the determinant has the following factorization*,

$$\det(\mathbf{A_n}) = R_0 R_1 R_2 \cdots R_{n-1}.$$

The mapping

$$(a_0, a_1, a_2, \cdots, a_{n-1}) \leftrightarrow (R_0, R_1, R_2, \cdots, R_{n-1})$$

is a discrete Fourier transformation. *Let* $a_0 = -x$, *we get the representation of roots for n-th order algebraic equation* $\det(\mathbf{A}_n) = 0$.

In the literatures of hypercomplex numbers, the multiplication rules of bases are in the following general form[17,20,30]

$$\boldsymbol{e}_j \boldsymbol{e}_k = C_{jk}^m \boldsymbol{e}_m, \qquad C_{jk}^m \in \mathbf{F}. \tag{12}$$

If the basis has inverse element $\boldsymbol{e}_k^{-1} = U^{jk} \boldsymbol{e}_j$ and satisfies

$$\boldsymbol{e}_k \boldsymbol{e}_k^{-1} = \boldsymbol{e}_k^{-1} \boldsymbol{e}_k = \boldsymbol{e}_0, \qquad (\boldsymbol{e}_0, \boldsymbol{e}_1^{-1}, \cdots, \boldsymbol{e}_{n-1}^{-1}) = \boldsymbol{e}U, \qquad |\det(U)| = 1. \tag{13}$$

Like Equation (10) we also denote matrices

$$\boldsymbol{C}^m = \frac{\partial \boldsymbol{M}}{\partial \boldsymbol{e}_m} = \left(C_{jk}^m\right), \qquad \boldsymbol{E}^m = \boldsymbol{C}^m (\boldsymbol{C}^0)^{-1}, \qquad \boldsymbol{A} = \boldsymbol{M}(\boldsymbol{C}^0)^{-1} = \boldsymbol{E}^m \boldsymbol{e}_m, \tag{14}$$

then we have

Theorem 3. *If the basic elements* $\{\mathbf{e}_k\}$ *satisfy Equations* (12–14), *then the associative algebra also satisfies the structure equation* $\mathbf{A}^2 = n\mathbf{A}$, *and the matrices* $\{\mathbf{E}^a\}$ *satisfy*

$$C_{ab}^c \boldsymbol{E}^a \boldsymbol{E}^b = n \boldsymbol{E}^c, \qquad (c = 0, 1, \cdots, n-1). \tag{15}$$

Proof. By Equation (13) we find

$$\boldsymbol{C^0} = (\boldsymbol{C^0})^{\boldsymbol{T}} = \boldsymbol{U^{-1}}, \qquad \boldsymbol{U} = (\boldsymbol{C^0})^{-\boldsymbol{1}}$$

is a symmetric matrix. Thus, the diagonal elements of matrix $(\boldsymbol{C}^0)^{-1}\boldsymbol{M} = (\boldsymbol{e}(\boldsymbol{C}^0)^{-1})^T\boldsymbol{e}$ are $\boldsymbol{e}_0\boldsymbol{I}$. By the associativity of multiplication, we have

$$\boldsymbol{A}^2 = \boldsymbol{e}^T(\boldsymbol{e}(\boldsymbol{C}^0)^{-1}\boldsymbol{e}^T)\boldsymbol{e}(\boldsymbol{C}^0)^{-1} = \boldsymbol{e}^T(n\boldsymbol{e}_0)\boldsymbol{e}(\boldsymbol{C}^0)^{-1} = n\boldsymbol{A} = n\boldsymbol{E}^c\boldsymbol{e}_c.$$

So the structure equation $\boldsymbol{A}^2 = n\boldsymbol{A}$ holds. On the other hand, we have

$$n\boldsymbol{E}^c\boldsymbol{e}_c = \boldsymbol{A}^2 = (\boldsymbol{E}^a\boldsymbol{e}_a)(\boldsymbol{E}^b\boldsymbol{e}_b) = \boldsymbol{E}^a\boldsymbol{E}^b(\boldsymbol{e}_a\boldsymbol{e}_b) = (C_{ab}^c\boldsymbol{E}^a\boldsymbol{E}^b)\boldsymbol{e}_c. \quad (16)$$

By Equation (16) and the linear independence of $\{\mathbf{e}_c\}$, we find Equation (15) holds. The proof is completed.

By Equation (14), for suitable structural coefficients C_{ab}^c, if the following map

$$\boldsymbol{E}_k \equiv (\boldsymbol{E}^k)^{-T} = (C^k U)^{-T} \leftrightarrow \boldsymbol{e}_k \quad (17)$$

is an isomorphism, and the norm of $\mathbf{X} = x^a\mathbf{E}_a$ is defined as Calvet's norm, then $\mathbf{x} \leftrightarrow \mathbf{X}$ meets the requirements of addition, subtraction, multiplication and division of the hypercomplex system. Unless the multiplication rule Equation (12) is reduced to Equation (5), for the general case $\{\boldsymbol{x} = x^a\boldsymbol{e}_a\}$, the basis elements are difficult to satisfy the isomorphic mapping Equation (17), so $\mathbf{x}$ is impossible to form a hypercomplex system. The above group-like properties should be the main feature of hypercomplex bases, in this case the basic elements are isomorphic to special matrices.

If $\{\mathbf{e}_a; a = 0, 1 \cdots n-1\}$ and $\{\mathbf{E}_b; b = 0,1 \cdots N-1\}$ are the canonical bases of two hypercomplex numbers, by the Kronecker product of the matrices $\mathbf{e}_a \otimes \mathbf{E}_b$, we obtain a composite hypercomplex system of *nN* unit elements

$$\boldsymbol{z} = z^{ab}\boldsymbol{e}_a \otimes \boldsymbol{E}_b, \qquad (0 \le a \le n-1, \qquad 0 \le b \le N-1). \quad (18)$$

For example, the above quaternion over $\mathbb{C}$ generated by Pauli matrices is equivalent to octonion $\mathbb{C} \otimes$ H. Thus, we can construct hypercomplex numbers with very complicated structure.

3. Clifford algebra as hypercomplex number

As a unified language of science, Clifford algebra has wide applications in geometry, physics, and engineering[31–38]. Clifford algebra has achieved good results in differential geometry, theoretical physics, classical analysis, and other aspects. The present author uses Clifford algebras in differential geometry[28] and unified field theory[39], and systematically explores the structure and fundamental properties of the hypercomplex system as well as its deep relationship with physical theories[25,29,40,41].

Recently, great progress has been made in the application of hypercomplex numbers in engineering. For example, in terms of image processing, in 1992, Ell proposed to use quaternion tracing to describe the RGB three-primary color model for each pixel[42]. In 1996, Sangwine used the hypercomplex Fourier transform for color processing[43]. These methods have the characteristics of simple programming and high data processing efficiency. The hypercomplex number has now been greatly developed and widely used in multi-channel information processing[44,45]. Hypercomplex numbers are also increasingly used in artificial intelligence, such as signal processing and deep learning[46,47]. Using the properties of Clifford algebra, the number of parameters can be significantly reduced while obtaining an efficient large-scale neural network model[48,49]. Quaternions are also used for precision control; for example, in 2000, Nadler et al. used quaternions for the iterative algorithm of GPS measurement to overcome the singularity problem when describing the coordinates of rigid body angle with Euler angle, and the quaternionic equation has the features of simple form and a small calculation amount[50]. In 2006, Tayebi and Mcgilvray used a quaternion as the feedback signal controlling the gyro moment of the vehicle[51].

Now we examine the relationship between Clifford algebra and hypercomplex numbers. For any vector in Minkowski space $\mathbf{x} = x^a \mathbf{e}_a \in \mathbb{R}^{p,q}$, we have product

$$\mathbf{x}^2 = x^a x^b (\mathbf{e}_a \mathbf{e}_b) = \frac{1}{2}(\mathbf{e}_a \mathbf{e}_b + \mathbf{e}_b \mathbf{e}_a) x^a x^b = \eta_{ab} x^a x^b \mathbf{I}, \tag{19}$$

where $(\eta_{ab}) = (\eta^{ab}) = \mathrm{diag}(\mathbf{I}_p, -\mathbf{I}_q)$ is Minkowski metric. For orthonormal basis vectors $\{\mathbf{e}_a\}$ and co-frames $\{\mathbf{e}^a = \eta^{ab} \mathbf{e}_b\}$, by Equation (19), the product of bases satisfies the Clifford relations Equation (2). The products $\mathbf{e}_a \mathbf{e}_b$ and $\mathbf{e}^a \mathbf{e}^b$ are called Clifford product or geometric product, and the algebra with geometric product is called Clifford algebra or geometric algebra, and denoted by $C\ell(\mathbb{R}^{p,q})$. In the case without confusion, we also use 1 to represent the unit matrix $\mathbf{I}$.

By Equation (19), we define the length of the vector as

$$|\mathbf{x}| = \sqrt{|\eta_{ab} x^a x^b|} = \sqrt{|x_a x^a|}.$$

For any two vectors $\mathbf{x} = x^a \mathbf{e}_a = x_a \mathbf{e}^a$ and $\mathbf{y} = y^a \mathbf{e}_a = y_a \mathbf{e}^a$, we have

$$\boldsymbol{xy} = \boldsymbol{x^a y^b} \left(\frac{1}{2}(\boldsymbol{e_a e_b} + \boldsymbol{e_b e_a}) + \frac{1}{2}(\boldsymbol{e_a e_b} - \boldsymbol{e_b e_a}) \right) = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \wedge \boldsymbol{y},$$

in which $\mathbf{x} \wedge \mathbf{y}$ is the exterior product of the vectors. The geometric meaning of the exterior product is the oriented area of the parallelogram constructed by the vectors x and y.

$$\boldsymbol{e_{ab}} = \boldsymbol{e_a} \wedge \boldsymbol{e_b} = \frac{1}{2}(\boldsymbol{e_a e_b} - \boldsymbol{e_b e_a}) = -\boldsymbol{e_{ba}}$$

forms the basis of an oriented area. $\mathbf{x} \cdot \mathbf{y} = \eta_{ab} x^a y^b$ is the inner product, and $\mathbf{x} \cdot \mathbf{y} = 0$ is call the two vectors are orthogonal. The basis set $\{\mathbf{e}_a\}$ satisfying Equation (2) is called orthonormal bases. Since Clifford algebra is isomorphic to special matrix algebra, the orthonormal basis vectors can be expressed by special square matrices, so that the geometric algebra transforms into the familiar matrix algebra. For the orthonormal basis vectors, Clifford products satisfy

$$\boldsymbol{e_j e_k} = -\boldsymbol{e_k e_j} = \boldsymbol{e_j} \wedge \boldsymbol{e_k}, \qquad (\boldsymbol{j} \neq \boldsymbol{k}).$$

In this case, the Clifford product and the exterior product of bases are equivalent. However, if the basis vectors are not orthogonal, the exterior product represents the directional volume of the parallelohedron, but it is not equivalent to the matrix product. On the contrary, the Clifford product has no geometric meaning but is equivalent to matrix multiplication. Therefore, the two products should be converted into each other in the computation. This is a subtle problem[26].

For the 1+3 dimensional realistic spacetime, the lowest-order complex matrix representation of the generators of Clifford algebra $C\ell(\mathbb{R}^{1,3})$ is Dirac-γ matrices

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 0 & \boldsymbol{I}_2 \\ \boldsymbol{I}_2 & 0 \end{pmatrix}, \qquad \gamma^a = -\gamma_a = \begin{pmatrix} 0 & -\sigma_a \\ \sigma_a & 0 \end{pmatrix},$$

which generate the Grassmann bases of $C\ell(\mathbb{R}^{1,3})$ as

$$\mathbf{I}_4, \qquad \gamma^a, \qquad \gamma^{ab}, \qquad \gamma^{abc} = -\epsilon^{abcd} \gamma_d \gamma^{0123}, \qquad \gamma^{0123} = -\mathrm{i}\gamma^5, \tag{20}$$

in which $\gamma^5 = \mathrm{diag}(\mathbf{I}_2, -\boldsymbol{I}_2), \epsilon^{0123} = 1$. We have the Clifford-Grassmann number as

$$\mathbf{K} = s\mathbf{I}_4 + A_a \gamma^a + H_{ab} \gamma^{ab} + Q_a \gamma^a \gamma^{0123} + p\gamma^{0123}, \tag{21}$$

where $(s, p, A_a, \cdots \in \mathbb{R})$. In the region $\{\det(\mathbf{K}) \neq 0\}$, the Clifford-Grassmann number Equation (21) is a hypercomplex number closed for addition, subtraction, multiplication and division, and it has $2^4 = 16$ dimension. We can also define the analytic functions for the hypercomplex numbers on the field $\mathbb{R}$, such as $\boldsymbol{H} = \boldsymbol{N}e^{\boldsymbol{W}}\sin(\omega\boldsymbol{T})\boldsymbol{A}^{-n}$, where $(\mathbf{H}, \mathbf{N}, \mathbf{W}, \cdots)$ are all Clifford-Grassmann numbers over $\mathbb{R}$. The matrix representation of the basis elements shows that Clifford algebra is equivalent to the matrix algebra expanded on the 2^n-dimensional Grassmannian bases Equation (20).

Clifford algebra has profound insights in description of physical laws[26,38,40,41]. We take the stationary fluid equations as example to show the application of geometric algebra and to reveal the hypercomplex structure of nonlinear physical equations[25]. The independent variables of the Newtonian fluid are density ρ and flow velocity $\mathbf{v}$, the pressure p is determined by equation of state $p = f(\rho, T, \cdots)$, and the unit volume force g is a known condition. For stationary fluid, all of these variables are functions of $\mathbf{x} \in \mathbb{R}^3$. In $C\ell(\mathbb{R}^{3,0})$, the generators $\{\sigma_a; a = 1,2,3\}$ satisfy the Clifford relation

$$\sigma_a\sigma_b + \sigma_b\sigma_a = 2\delta_{ab}, \qquad \sigma_\mu\sigma_\nu + \sigma_\nu\sigma_\mu = 2g_{\mu\nu}, \qquad \sigma^a = \delta^{ab}\sigma_b, \tag{22}$$

where σ^a's is Pauli matrices Equation (7). We have the Grassmann basis elements

$$\mathbf{I} \in \Lambda^0, \qquad \sigma_a \in \Lambda^1, \qquad \sigma_{ab} \equiv \sigma_a \wedge \sigma_b = \epsilon_{abc}\mathrm{i}\sigma^c \in \Lambda^2, \qquad \sigma_{abc} = \epsilon_{abc}\mathrm{i}\mathbf{I} \in \Lambda^3. \tag{23}$$

From the point of view of hypercomplex numbers,

$$d\boldsymbol{x} = dx^a\sigma_a, \qquad \boldsymbol{v} = v^a\sigma_a, \qquad \nabla = \sigma^a\partial_a, \qquad (a = 1,2,3)$$

are all Λ^1 numbers in $C\ell(\mathbb{R}^{3,0})$, and $\rho = \rho\mathbf{I} \in \Lambda^0$ is a scalar.

Theorem 4. *The stationary flow of Newtonian fluid satisfies the following hypercomplex number equations*

$$\frac{1}{2}[\nabla(\rho\mathbf{v}) + (\nabla(\rho\mathbf{v}))^+] = 0, \tag{24}$$

$$\frac{1}{2}[\boldsymbol{v}\nabla\boldsymbol{v} - \boldsymbol{v}(\nabla\boldsymbol{v})^+ + \nabla\boldsymbol{v}^2] = \boldsymbol{T} - \rho^{-1}\nabla p + \boldsymbol{g}, \tag{25}$$

in which

$$\boldsymbol{T} = \nu\nabla^2\boldsymbol{v} + \frac{1}{2}\mu\nabla(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^+)$$

is viscous force, $(\mu \geq 0, \nu \geq 0)$ *are viscosity coefficients. The velocity v satisfies the following consistent equation*

$$\vec{v} \cdot (\mathrm{curl}\,\vec{v}) = v_1(\partial_2 v_3 - \partial_3 v_2) + v_2(\partial_3 v_1 - \partial_1 v_3) + v_3(\partial_1 v_2 - \partial_2 v_1) = 0. \tag{26}$$

Proof. By expanding Equation (24) and using Clifford relation Equation (22), we have

$$\frac{1}{2}\partial_a(\rho v_b)(\sigma^a\sigma^b + \sigma^b\sigma^a) = \partial_a(\rho v^a) = 0.$$

The above equation is the continuity equation $\mathrm{div}(\rho\vec{v}) = 0$.

According to Clifford algebra, the nonlinear terms in Equation (25) belong to the $\Lambda^1 \cup \Lambda^3$. Projecting Equation (25) onto the Grassmann bases Equation (23), then the Λ^1 terms give the equation of motion

$$v^b\partial_b v^a = \nu\Delta v^a + \mu\partial_a(\mathrm{div}\,\vec{v}) - \rho^{-1}\partial_a p + g_a, \tag{27}$$

and the Λ^3 terms are a pseudo scalar, which give the consistent Equation (26). The proof is completed.

For the stationary fluid, the speed should satisfy the consistent Equation (26). Therefore, the Navier-Stokes equation may need other constraints to determine the solution. The Navier-Stokes equation satisfies neither Clifford algebra $C\ell(\mathbb{R}^{3,0})$ nor $C\ell(\mathbb{R}^{1,3})$, thus it is flawed in algebra. The above discussion shows that the description of nonlinear phenomena only by vector algebra is incomplete, ignoring the order and solvability of physical quantities. Only expressed by hypercomplex numbers, can the certainty of solution of physical equations can be guaranteed, otherwise the equations may be contradictory or undetermined.

4. Discussion and conclusion

From the development course of hypercomplex numbers, we learn that the construction of mathematical theory is quite arbitrary. Mathematics is similar to a sea without boundaries, and any set of logically compatible relations and propositions can be regarded as a mathematical theory. However, only the mathematical theories suitable for describing the laws of nature are the best and the simplest ones, so we should learn math from nature. From this point of view, we can define that a hypercomplex number is a finite dimensional vector space over field F, and the basis vectors satisfy the group-like properties. Natural laws are high-dimensional and nonlinear and therefore should be described by hypercomplex numbers. Although vector algebra is also a good tool to describe high-dimensional variables, it still lacks the operation of numbers and does not define division operations, so it is difficult to adapt to the nonlinear relations of complicated systems. According to the Frobenius theorem, $\mathbb{R}, \mathbb{C}, \mathrm{H}$ are the only finite-dimensional associative division algebras over $\mathbb{R}$ without zero factor. If the multiplication associativity is abandoned again, the division algebra without zero factor remains only octonions or Cayley numbers. The main reasons limiting the expansion of number systems are the universal definition of norm $||\cdot||$ and the zero factor condition $||\mathbf{a}|| = 0 \Leftrightarrow \mathbf{a} = 0$, while the zero factor condition has little effect on the algebraic operation and application of hypercomplex numbers. If this limitation is relaxed, then we can construct a large number of hypercomplex systems with high application value.

Hypercomplex numbers are isomorphic to some special matrix algebras, with addition, subtraction, multiplication, and division operations, and meet distributivity and associativity, so they are easy to learn and easy to use. This paper briefly introduces several basic properties and application examples of hypercomplex numbers, and we can see from the discussion that physical quantities have a structure of hypercomplex numbers. The hypercomplex number system combines the advantages of mathematical tools such as algebra, geometry, and analysis, and its orthonormal bases form a group-like associative algebra that satisfies the structural equation $\mathbf{A}^2 = n\mathbf{A}$. The zero norm set of hypercomplex numbers has special geometric significance, reflecting the intrinsic properties of algebra and fundamental space, so it is worthy of further study. The existence of a zero norm set will have little influence on algebraic operation, which is far from the serious problem of giving up associativity. The hypercomplex number is simple and intuitive and has wide applications, so it is worthy of learning in basic education and application in scientific research and engineering technology.

Conflict of interest

The author declares that he has no conflict of interest.

References

1. Kline M. *Mathematical Thought from Ancient to Modern Times*. Oxford University Press; 1972.
2. Sun Z. After the plural—How far can we go (Chinese). *Chinese Journal of Nature* 1992; 15(8): 615–619.
3. Hamilton WR. Theory of quaternions. *Proceedings of the Royal Irish Academy (1836-1869)* 1844; 3: 1–6.
4. Frobenius H. On linear substitutions and bilinear forms (German). *Journal für die Reine und Angewandte Mathematik (Crelles Journal)* 1878; 1878(84): 1–63. doi: 10.1515/crelle-1878-18788403
5. Hurwitz A. About the composition of the square shapes (German). *Mathematische Annalen* 1922; 88: 1–25. doi: 10.1007/BF01448439

6. Milnor J. Some consequences of a theorem of Bott. *Annals of Mathematics* 1958; 68(2): 444–449. doi: 10.2307/1970255
7. Gibbs JW. *Elements of Vector Analysis: Arranged for the Use of Students in Physics*. Tuttle, Morehouse & Taylor; 1884.
8. Heaviside O. Electromagnetic induction and its propagation. *The Electrician* 1885; 14: 178–180.
9. Grassmann HG. *The Linear Expansion Theory* (German). Verlag von Otto Wigand; 1844.
10. Clifford WK. Applications of Grassmann's extensive algebra. *American Journal of Mathematics* 1878; 1(4): 350–358. doi: 10.2307/2369379
11. Clifford WK. On the classification of geometric algebras. In: Tucher R (editor). *Mathematical Papers*. Macmillian; 1882. pp. 397–401.
12. Peirce B. Linear associative algebra. *American Journal of Mathematics* 1881; 4(1): 97–229. doi: 10.2307/2369153
13. Study E. About systems of complex numbers (German). In: *News from the Royal Society of Sciences and the Georg August University of Göttingen* (German). Dieterich'sche Verlagsbuchhandlung; 1889. pp. 237–268.
14. Study E. *The Geometry of Dynamism* (German). Druck und Verlag von B. G. Teubner; 1903.
15. Kantor IL, Solodovnikov AS. In: Shenitzer A (translator). *Hypercomplex Numbers: An Elementary Introduction to Algebras*, 1st ed. Springer New York; 1989.
16. Cayley Esq A. On the theory of groups, as depending on the symbolic equation $\theta^n = 1$. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 1854; 7(42): 40–47. doi: 10.1080/14786445408647421
17. Molien T. About systems of higher complex numbers (German). *Mathematische Annalen* 1892; 41: 83–156. doi: 10.1007/BF01443450
18. Molien T. A note on the theory of homogeneous substitution groups (German). S*itzungsberichte der Naturforscher-Gesellschaft Dorpat* 1897; 11: 259–274.
19. Frobenius G. *On the Representation of Finite Groups by Linear Transformations* (German). Preussischen Akademie; 1897.
20. Cartan E. Bilinear groups and complex number systems (French). *Annales de la Facultė des Sciences de Toulouse: Mathėmatiques* 1898; 12(1): B1–B64.
21. van der Waerden BL. Chapter 10, 11, 13. In: *A History of Algebra: From al-Khwārizmī to Emmy Noether*, 1st ed. Springer Berlin; 1985. doi: 10.1007/978-3-642-51599-6
22. Wedderburn JM. On hypercomplex numbers. *Proceedings of the London Mathematical Society* 1908; 2(1): 77–118.
23. Dickson LE. On hypercomplex number systems. *Transactions of the American Mathematical Society* 1905; 6(3): 344–348.
24. Dickson LE. On quaternions and their generalization and the history of the eight square theorem. *Annals of Mathematics* 1919; 20(3): 155–171. doi: 10.2307/1967865
25. Gu Y. Hypercomplex numbers and roots of algebraic equation. *Journal of Geometry and Symmetry in Physics* 2022; 64: 9–22. doi: 10.7546/jgsp-64-2022-9-22
26. Gu Y. Clifford algebras, hypercomplex numbers and nonlinear equations in physics. *Geometry, Integrability and Quantization* 2023; 25: 47–72. doi: 10.7546/giq-25-2023-47-72
27. Calvet RG. On matrix representations of geometric (Clifford) algebras. *Journal of Geometry and Symmetry in Physics* 2017; 43: 1–36. doi: 10.7546/jgsp-43-2017-1-36
28. Gu Y. A note on the representation of Clifford algebras. *Journal of Geometry and Symmetry in Physics* 2021; 62: 29–52. doi: 10.7546/jgsp-62-2021-29-52
29. Gu Y. Clifford algebra and hypercomplex number as well as their applications in physics. *Journal of Applied Mathematics and Physics* 2022; 10(4): 1375–1393. doi: 10.4236/jamp.2022.104097
30. Hawkins T. Hypercomplex numbers, lie groups, and the creation of group representation theory. *Archive for History of Exact Sciences* 1972; 8: 243–287. doi: 10.1007/BF00328434
31. Lounesto P. *Clifford Algebras and Spinors*, 2nd ed. Cambridge University Press; 2001. pp. 25–37.
32. Hestenes D. Spacetime physics with geometric algebra. *American Journal of Physics* 2003; 71(7): 691–714. doi: 10.1119/1.1571836
33. Doran C, Lasenby A. *Geometric Algebra for Physicists*. Cambridge University Press; 2003.
34. Ablamowicz R, Baylis WE, Branson T, et al. In: Ablamowicz R, Sobczyk G (editors). *Lectures on Clifford (Geometric) Algebras and Applications*. Birkhäuser; 2004.
35. Ablmowicz R. *Clifford Algebras: Applications to Mathematics, Physics, and Engineering*. Birkhäuser; 2004.
36. Lundholm D, Svensson L. Clifford algebra, geometric algebra, and applications. Available online: https://www.mathematik.uni-muenchen.de/~lundholm/clifford.pdf (accessed on 21 September 2023).
37. Perwass C, Edelsbrunner H, Kobbelt L, Polthier K. *Geometric Algebra with Applications in Engineering*. Springer Berlin; 2009. doi: 10.1007/978-3-540-89068-3
38. Dressel J, Bliokh KY, Nori F. Spacetime algebra as a powerful tool for electromagnetism. *Physics Reports* 2015;

589: 1–71. doi: 10.1016/j.physrep.2015.06.001
39. Gu Y. *Clifford Algebra and Unified Field Theory*. LAP LAMBERT Academic Publishing; 2020.
40. Gu Y. Hypercomplex numbers and the origin of celestial magnetic fields. In: Proceedings of the ATINER's Conference; 13 September 2023; Athens, Greece.
41. Gu Y. Spinor equation and operator algebra. *Journal of Geometry and Symmetry in Physics* 2023; 66: 1–34.
42. Ell TA. *Hypercomplex Spectral Transformations* [PhD thesis]. Minnesota University; 1992.
43. Sangwine SJ. Fourier transforms of colour images using quaternion or hypercomplex, numbers. *Electronics Letters* 1996; 32(21): 1979–1980. doi: 10.1049/el:19961331
44. Lan X, Liu W. Fully quaternion-valued adaptive beamforming based on crossed-dipole arrays. *Electronics* 2017; 6(2): 34. doi: 10.3390/electronics6020034
45. Took CC, Xia Y. Multichannel quaternion least mean square algorithm. In: Proceedings of the 2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP); 12–17 May 2019; Brighton, UK. pp. 8524–8527.
46. Xiang M, Enshaeifar S, Stott AE, et al. Simultaneous diagonalisation of the covariance and complementary covariance matrices in quaternion widely linear signal processing. *Signal Processing* 2018; 148: 193–204. doi: 10.1016/j.sigpro.2018.02.018
47. Liu Y, Zhang D, Lou J, et al. Stability analysis of quaternion-valued neural networks: Decomposition and direct approaches. *IEEE Transactions on Neural Networks & Learning Systems* 2018; 29(9): 4201–4211. doi: 10.1109/TNNLS.2017.2755697
48. Parcollet T, Morchid M, Linares G, Mori RD. Bidirectional quaternion long short-term memory recurrent neural networks for speech recognition. In: Proceedings of the 2019 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP); 12–17 May 2019; Brighton, UK. pp. 8519–8523.
49. Valle ME, Lobo RA. Hypercomplex-valued recurrent correlation neural networks. *Neurocomputing* 2021; 432(7): 111–123. doi: 10.1016/j.neucom.2020.12.034
50. Nadler A, Bar-Itzhack IY, Weiss H. On the algorithms for attitude estimation using GPS. In: Proceeding of the 39th IEEE Conference on Decision and Control; 12–15 December 2015; Sydney, Australia. pp. 3321–3326.
51. Tayebi A, Mcgilvray S. Attitude stabilization of a VTOL quadrotor aircraft. *IEEE Transactions on Control System Technology* 2006; 14(3): 562–571. doi: 10.1109/TCST.2006.872519