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Review

The biodiversity interface fractional diffusion problem: possible use of fractional calculus and legendre polynomials

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Abstract: In this paper, the classic reaction-diffusion and chemotaxis models are compared with the mathematical techniques of fractional calculus, with special reference to the evolvements at a biodiversity border interface. The fractional diffusion-wave approach of Mainardi, methods of orthogonal polynomials and possible alternatives for the Dirichlet– Neumann boundary condition are reviewed. The evolvement of ecosystems at a border interface follows a combination of at least two types of causal relationships, one of a stochastic-deterministic nature and another following a hereditary stepping-stone characteristic, similar to the Fibonacci recurrence relation. It has been suggested that the use of shifted Legendre polynomials contribute to a conceptual framework for describing the hereditary heuristic in materials science, which is confirmed in this paper in an ecological setting. However, further mathematical proof of this conjecture (of a hereditary heuristic) remains to be solved in future work. Also, foraging-sheltering dipoles offer an interesting application of the dipole notion in mathematical mapping theory. Examples are for instance found in the ecosystem development in the tidal-littoral zone and in its characteristic foraging and nesting bird populations. Also here, further sophisticated mathematical tools and ecological assessment procedures need further work to make this compelling ideas tangible for hard analysis.

Keywords: Fractional calculus; fractional diffusion at border interface; Legendre polynomials; Dirichlet – Neumann boundary condition; memory and Fibonacci recurrence relation.

1. Introduction

The notion of Fractional Diffusion gained new interest as a merger of Fractional Calculus [1] and Diffusion-Wave Equations [2] in a fractal geometry [3]. One of the first descriptions of transfer processes in fractals, in the sense of Benoit B. Mandelbrot (1924-2010), was given by Le Mehaute and co-workers [4], suggesting an equation of the form:

$$ {}_0D_t^{\frac{1}{d-1}} J(t) = LX(t) \tag{1} $$

where $J(t)$ is the macroscopic flow across a fractal interface, $X(t)$ is the local driving force, L is a constant and d is the fractional dimension. These transport processes in fractal media were found to have interesting applications in the fields of electro-chemistry, viscoelasticity, crystallo-genesis and chemistry of porous materials [4,5]. The fractional dimension of multi-trophic ecosystems and characteristics of biodiversity resilience form a challenging application of fractional calculus and the mathematics of dynamic ecosystems [6,7].

The origins of Fractional Calculus (FC), the theory of derivatives (and integrals) of non-integer order, go back more than three centuries: namely, in a note of G.W. Leibniz (1646-1716) reflecting on the work of the Marquis de l'Hôpital (1661-1704),

Leibniz suggested a meaning for the derivative of order one half [8]. The notation used for the fractional derivative is ${}_aD_t^{\alpha} f(t)$, with a and t the limits of the operation, and α the order of fractional differentiation. Similarly, the notation ${}_aD_t^{-\beta} f(t)$ is used for a fractional integration with order $-\beta$, where β is a non-integer positive number. Fractional differential equations have proven to be useful for numerical solutions of engineering, nanoparticle physics, astronomy, neurobiology and cybernetics [1]. An important development to FC was made by M. Caputo (° 1927). Namely, the Caputo approach allows "the formulation of initial conditions for initial-value problems for fractional order differential equations in a form involving only the limit values of integer-order derivatives at the lower terminal (initial time)" [1]. The problem of fixed boundary conditions (for ordinary and partial differential equations) is also known as the Dirichlet problem [9]. It will appear to have important consequences for our study of fractal ecosystems and will be further discussed in Section ¶ **3.2. Use of (Shifted) Legendre Polynomials and possible alternatives for the Dirichlet Boundary Condition.**

The translation of the boundary value problem in a mathematical correspondence of the ecological biodiversity problem, translated as a Probability Density Function (PDF), is one of the core problems addressed in this study. Podlubny [1] already referred to a fractional partial differential equation as a replacement of the classical version of A.E. Fick's law (1829-1901). This application of FC was first suggested by Oldham and Spanier [10], and worked out by Giona and Roman [11]. In its simplest form, this resulted in the following equation:

$$ {}_0D_t^{\frac{1}{d}} P(r,t) = -A\left(\frac{\partial P(r,t)}{\partial r} + \frac{\kappa}{r} P(r,t)\right) \tag{2} $$

where $P(r,t)$ is the average probability density of random walks on fractals, A and κ are constants; and, d is the anomalous diffusion exponent, which depends on the fractal dimension of the considered media [12] . Regarding the fractal appearance of ecosystem development, moreover, the lattices where fractional diffusion-wave phenomena take place are in itself the result of a fractal generating mechanism. The dispersion of species according to these apparent 'stochastic' fractal pathways, which form the primary 'border line' for species migration, then becomes a secondary fractal system. On the other hand, the correspondence between a biodiversity index and a PDF based on the occupancy of niches in an ecological phase space, identifiable with a set of dimensions of fractal nature and de-composed as a (number of) foraging-sheltering dipoles [6], shows an analogy to the use of continuous time random walk (CTRW) models (with jump length depending on waiting time) on a lattice [13].

Moreover, the method of fractional integrals was found an insightful instrument to describe a kind of memory, like for instance in hereditary changes of mechanical properties of steel wires [1,14]. Memory mechanisms as these, forward the question of the use of causal functions, and also how causal functions could be explained in a biological or ecological context. For individuals of vertebrates and other highly developed animal species, obviously, memory is stored in the functioning of the brains. However, for the biodiversity of ecosystems, memory should be defined in a supra-individualistic and even beyond a species-specific causal heuristic (See ¶ **4. Heuristics for Resolving the 'Memory function' in Biodiversity Resilience**).

Mathematically, causal functions allow for the implementation of recursive functions and their convolution through the Laplace transform [1,15]. The method of least squares allows the fitting with physical empirical data and may represent the cumulative impact of previous history on the actual performance of a system.

Moreover, solutions for non-homogeneous boundary conditions, such as in the Liouville-Caputo fractional Bagley-Torvik differential problem [16], lead into the use of (shifted) Legendre polynomials [16]. Besides, a solution that does not require differentiation, like in the problem of N.H. Abel's (1802-1829) integral equation [17], is suggestive for the use of the method of orthogonal polynomials, as was explained in his seminal work by Podlubny [1]. The possible use of Legendre polynomials, Chebyshev polynomials and other orthogonal polynomial methods in the fractional description of ecosystems is discussed in Section ¶ **2.3. The Caputo Approach and Methods of Orthogonal Polynomials**).

The main structure of this paper consists of three parts: in Section ¶ **2. Mathematical Background** the basics of reaction-diffusion and chemotaxis models are summarized, as well as an introduction to Fractional Calculus and the use of Orthogonal Polynomials. An important emphasis is laid upon the abstract presuppositions followed in classic reaction-diffusion theory and how these presuppositions correspond with biological entities like species and multi-species ecosystems. In Section ¶ **3. Application to Fractal Biological and Ecological Problems**, a review is presented of the available knowledge on the application of the previous mathematical methods to the realm of fractal ecosystems and the biodiversity interface problem. In Section ¶ **4. Heuristics for Resolving the 'Memory function' in Biodiversity Resilience**, we propose some possible heuristics for the translation of the memory and resilience characteristics of ecosystems in a mathematical mapping theory. Finally, in Section ¶ **5.** support from ecological observation data is presented.

2. Mathematical Background

2.1. Classic Reaction-Diffusion and Chemotaxis models

According to J. D. Murray [18], the process of random walk forms a well-known heuristic for the derivation of the diffusion equation. For in one dimension, "consider the probability p (m, n) that a particle reaches a point m space steps to the right (that is to $x = m\Delta$x) after n time steps (after a time $n\Delta t$), and supposing that to reach $m\Delta x$, the particle has moved a steps to the right and b steps to the left, then $m = a - b$ and $a+b = n$, or, $a = ½ (n+m)$ and $b = n – a$". [18] Then, the number of possible paths that a particle can reach this point $x = m\Delta x$ equals $n!/(a\ !\ b!) = n!/\ (a!\ (n\text{-}a)!)$, which is the binomial coefficient C_a^n, defined for example by the expression [18]:

$$(x+y)^n = \sum_{a=0}^{n} C_a^n x^{n-a} y^a \tag{3}$$

The total number of possible n steps is 2^n. So, the probability p (m,n) representing the ratio of favorable to total possibilities, is then given by

$$p(m,n) = \frac{1}{2^n} \cdot \frac{n!}{a!(n-a)!} \tag{4}$$

noting also that the sum of all probabilities must equal unity, or :

$$\sum_{m=-n}^{n} p(m,n) = 1 \tag{5}$$

Mathematically, this is fulfilled, since [18]:

$$\sum_{m=-n}^{n} p(m,n) = \sum_{a=0}^{n} C_a^n (\frac{1}{2})^{n-a} (\frac{1}{2})^a = (\frac{1}{2} + \frac{1}{2})^n = 1 \tag{6}$$

If one let n become large, so that also $n + m$ be large, then one obtains James Stirling's (1692-1770) formula for the gamma function (Γ), which is an asymptotic approximation for factorials. Namely:

$$n! \sim (2\pi n)^{\frac{1}{2}} n^n e^{-n} \qquad \text{for} \qquad n \to \infty \tag{7}$$

known as Stirling's formula, which in fact was first discovered by Abraham de Moivre in 1721 [19]:

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t} t^n \mathrm{d}t \tag{8}$$

Meanwhile, it is important to note here that the gamma function, or $\Gamma(n) = (n–1)!$, is an extension of the factorial function to all complex numbers, except for negative integers and zero, where it is not defined (for application see e.g. in [20]).

From the latter formula, according to [18], the normal or Gaussian probability distribution is obtained:

$$p(m,n) \sim \left[\frac{2}{\pi n}\right]^{\frac{1}{2}} \cdot \exp\left[\frac{-m^2}{2n}\right] \qquad \text{for} \qquad m \geq 1, n \geq 1 \tag{9}$$

When taking the limits of $m \to \infty$ and $n \to \infty$, and taking into account that the probability of finding a particle at a certain point x must tend to zero, as the number of points tend to ∞ , then also the limits to zero for Δx and Δt are prescribed. Or, the previous expression (9) becomes [18]:

$$\frac{p(\frac{x}{\Delta x}, \frac{t}{\Delta t})}{2\Delta x} \sim \left(\frac{\Delta t}{2\pi t(\Delta x)^2}\right)^{\frac{1}{2}} \cdot \exp\left(\frac{-x^2}{2t} \cdot \frac{\Delta t}{(\Delta x)^2}\right) \tag{10}$$

If one assumes that the double limit:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{(\Delta x)^2}{2\Delta t} \to D \neq 0 \tag{11}$$

then the expression transforms in the diffusion equation, with D the diffusion coefficient or diffusivity of the particle [18]:

$$u(x,t) = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{p(\frac{x}{\Delta x}, \frac{t}{\Delta t})}{2\Delta x} = \left(\frac{1}{4\pi Dt}\right)^{\frac{1}{2}} \cdot \exp\left(\frac{-x^2}{4Dt}\right) \tag{12}$$

According to Murray [18], the latter result corresponds with the classical approach to diffusion governed by Fick's law, also known as 'Fickian' diffusion. In one dimension, the law is represented by the following expression:

$J = - D \partial C / \partial x$ with J the flux of material, C (x, t) the concentration of the (particle) species and D its diffusivity. For Murray, this 'material' could be everything or anything, like cells, chemical molecules, but also the number of animals in a given area.

Evidently, migrating animals, plants or biodiversity indices [6], do not behave like randomly hopping particles on a lattice, for biological species are in need of, and recognize specific cues for food, shelter and abiotic parameters. In other words, the criticism is justified that the dynamics of biodiversity are as far as possible removed from the idea of putting a jar filled with bees in an area, that is unfamiliar to the bees; and, opening the jar, resulting in a cloud of stressed and disarranged bees, is in no way similar to the behavior of a wild bee colony, neither of any other social insect species [15]. So, in order to make the diffusion methodology applicable to an ecological system, at least some mechanisms of (chemical) attraction as well as

physical (or behavioral) repulsion are needed. In the classic approach of Murray [18], chemical attraction is added to the mechanism of 'purely' random walk.

For an isotropic random walk without any 'bias', taking advantage of the use of Paul Dirac's (1902-1984) delta function (δ) for the release of Q particles per unit area at $x = 0$ and $t = 0$ [21], analogous to the opening of the jar full of bees at $t = 0$, Murray obtains the following expression for the probability distribution:

$$\frac{\partial p}{\partial t} = \left(\frac{(\Delta x)^2}{2\Delta t}\right)\frac{\partial^2 p}{\partial x^2} + \left(\frac{\Delta t}{2}\right)\frac{\partial^2 p}{\partial t^2} + \cdots \tag{13}$$

and, with $C(x, 0) = Q\,\delta(x)$ as an initial (boundary) value, the following solution of Fick's diffusion for $t > 0$ is given [18]:

$$C(x,t) = \frac{Q}{(4\pi Dt)^{\frac{1}{2}}} \cdot e^{\frac{-x^2}{4Dt}} \tag{14}$$

Supposedly, adding the presence of a gradient of a (chemical) attractant a (x, t), according to Murray [18], the flux of materials (cells, individuals of an animal species, plants, et cetera) will increase with the number of materials present, or n (x, t). This results in the following expression for the chemotactic flux:

$$J = n\,\chi\,(a)\,\nabla\,a \tag{15}$$

with χ(a) the attractant concentration function, and ∇ (a) the (spatial) divergence. Furthermore, following the conservation equation for the 'growth' of materials:

$$f(n) = \partial\, n\, /\partial\, t\, \nabla\, .\, J \tag{16}$$

and, assuming: $J = J_{diffusion} + J_{chemotaxis}$

the following expression is obtained for the (population) reaction–diffusion–chemotaxis equation [18]:

$$\frac{\partial n}{\partial t} = f(n) - (\nabla \cdot n\chi(a)\nabla a) + \nabla \cdot D\nabla n \tag{17}$$

For instance, in one spatial dimension, a system of nonlinear differential equations has been obtained for a bacterial species n (or an amoeba) that feeds on a chemical attractant a (see e.g. [22]):

$$\begin{aligned} \frac{\partial n}{\partial t} &= D\frac{\partial^2 n}{\partial x^2} - \chi_0 \frac{\partial}{\partial x}\left(n\frac{\partial a}{\partial x}\right) \\ \frac{\partial a}{\partial t} &= hn - ka + D_a\frac{\partial^2 a}{\partial x^2} \end{aligned} \tag{18}$$

where D_a is the diffusivity of chemical a, h and k are positive constants representing the production kinetics and consumption of a (a being neglectable). Examples are found in the kingdom of bacteria, amoeba and slime molds, where the attractants are diffusible chemicals produced by the organisms and where no higher brain functions are expected. For the organisms that are envisaged in biodiversity studies, at least also a repulsive mechanism is needed: since animals not only are positively attracted by chemicals or food provided by the presence of human settlements, but, more importantly, (higher) animals mostly evade the presence of human disturbance [7].

Alternatively, in the so-called 'diffusing diffusivity approach' (see e.g. Yuicho Itto [2019]), the distribution of diffusivities of RNA-protein particles is proposed as a model for a system with non-constant, fluctuating diffusivity [23]. Analogously, diffusing diffusivity could be invoked for the fluctuations of the diffusivity of individual species too, with changing composition of the species-pool or biodiversity.

In this analogy, the diffusivity of species could be described by exponentials, whereas the fluctuations of biodiversity would follow a power law, following the time required for a species to adapt to a change in local biodiversity [23]. So far, we have not followed this line of inquiry, because it does not discriminate between the various kinds of ecological dispersion, be it by diffusion, chemotaxis, repulsion or by wave propagation.

2.2. Some basics of Fractional Calculus

Starting from the well-known definitions of the first, second, third-order derivatives, $f'(t), f''(t), f'''(t)$, et cetera:

$$f'(t) = \frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{h\to 0} \frac{(f(t) - f(t-h))}{h} \tag{19}$$

$$f''(t) = \frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = \lim_{h\to 0} \frac{(f'(t) - f'(t-h))}{h} = \lim_{h\to 0} \frac{(f(t) - 2f(t-h) + f(t-2h))}{h} \tag{20}$$

by induction, a general, integer (n)-order derivative is obtained, using the expression (^n_r) for the binomial coefficient (or $n\ !/\ (r!(n\text{-}r)!)$ [1]:

$$f^{(n)}(t) = \frac{\mathrm{d}^n f}{\mathrm{d}t^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} (t - rh) \tag{21}$$

In a similar, but rather more sophisticated approach [1], the general expression for integer (p)-order integration is given by:

$${}_aD_t^{(-p)} f(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} h^p \sum_{r=0}^{n} \binom{p}{r} f(t - rh) = \frac{1}{(p-1)!} \int_a^t (t-\tau)^{p-1} f(r)\mathrm{d}\tau \tag{22}$$

The latter general expression for integer-order integration follows from the expression with binomial coefficients, whereby a negative value for p is introduced (commonly square instead of round brackets are used). Moreover, in order to arrive at a non-zero limit as $h \to 0$, the substitution $h = (t - a)/n$ is made, with a being a real constant. Then the limit value, either finite or infinite, is denoted by the following expression for the operation of $f(t)$, a and t being the 'terminals' related to the operation:

$$\lim_{\substack{h\to 0 \\ nh=t-a}} f_h^{(-p)}(t) =_a D_t^{(-p)} f(t) \tag{23}$$

Then, in the Riemann – Liouville approach for fractional derivatives [1], an extension is provided for the differentiation to a non-integer order. Namely, in the expression of integer order $k - n$, the integer n is substituted by a real number α, with $k - \alpha > 0$, giving:

$${}_aD_t^{(k-\alpha)} f(t) = \frac{1}{\Gamma(\alpha)} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau \tag{24}$$

with $\Gamma(\alpha)$ the gamma function with real-valued argument, as in Stirling's formula (see eg. [8]).

2.3. The Caputo Approach and Methods of Orthogonal Polynomials

The importance of Caputo's contribution to FC , according to [1], consisted of

solving the fractional derivative or - integral equation by finding the limit values of integer-order derivatives at the lower terminal (initial time or begin value problem [BVP]) of generalized functions. Accordingly, the use of the Dirac delta function (δ), or, Oliver Heaviside's (1850-1925) (unit-step) function H (t), and others are frequently used in applications of FC in electric engineering and telecommunication models [1].

Understanding the rationale of the method of orthogonal polynomials, for instance may follow from the problem known as the solution of Niels H. Abel's (1802-1829) integral equation [1]:

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{y(\tau)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau = f(t) \qquad (0<t<1) \tag{25}$$

Following the saying that, according to [24], "the solution of the integral equation (of Abel) of the first kind is an ill-posed problem", because the differentiation of this equation leads to "magnification of noise in measured data and to wrong numerical results", an alternative approach is needed [1]. To circumvent this problem, the method of orthogonal polynomials is proposed to the solution of fractional integral equations: e.g. the C.G.J. Jacobi (1804-1851) polynomials, or the P.L. Chebyshev (1821-1894) and A.M. Legendre (1752-1833) polynomials. The method of orthogonal polynomials was (allegedly) originally developed by G. Ya. Popov [25], but also other sources have been named [26].

Taking for instance the solution which is unbounded at t = +/- 1, using the spectral relationship between Chebyshev polynomial of the first kind (T_n (t)) and Chebyshev polynomials of the 2nd kind (U_n (t)), namely:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(\tau)\mathrm{d}\tau}{(\tau-t)\sqrt{(1-\tau^2)}} = U_{n-1}(t) \qquad \text{with} -1<t<1 \qquad \text{and} \qquad n=1,2,\cdots \tag{26}$$

a solution of Abel's integral equation (see above), according to [1], is possible for the solution of y (t) in the form of a series of Chebyshev polynomials of the 1st kind, where the coefficients y_0 have to be found:

$$y(t) = \frac{1}{\sqrt{1-t^2}}\sum_{n=0}^{\infty} y_n T_n(t) \tag{27}$$

Moreover, taking advantage of the known integral

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\mathrm{d}\tau}{(\tau-t)\sqrt{1-\tau^2}} = 0 \tag{28}$$

and, taking into account the orthogonality of the Chebyshev polynomials of the 2nd kind, or:

$$\int_{-1}^{1} U_k(t)U_m(t)dt = \begin{cases} \frac{\pi}{2} & \text{for } k=m, \\ 0 & \text{for } k\neq m. \end{cases} \tag{29}$$

Podlubny [1] obtains:

$$y_n = \frac{2}{\pi}\int_{-1}^{1} f(t)U_{n-1}(t)\sqrt{1-t^2}\mathrm{d}t \tag{30}$$

with n = 1, 2,… , and application of the additional condition (e.g. A) providing the uniqueness of the solution [1], the value of y_0 is obtained from the expression:

$$A = \int_{-1}^{1} y(t)\mathrm{d}t = \sum_{n=0}^{\infty} y_n \int_{-1}^{1} \frac{T_n(t)\mathrm{d}t}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} y_n \int_{-1}^{1} \frac{T_n(t)T_0(t)\mathrm{d}t}{\sqrt{1-t^2}} = \pi y_0$$

(31)

Also other spectral relationships may be used for solving the Riemann-Liouville fractional integrals (and fractional differentials), such as the Legendre polynomials (although not always providing an orthogonal basis), following the use of finite J. Fourier (1768-1830) transforms [27]. Interestingly, in the group of M. Abdelhakem [28,29], the use of shifted Legendre polynomials (SLP) was introduced for the purpose of spectral analysis of a fractional-differential (resp. - integral) function over the interval [0, 1]. Doing so, and making use of the shifted Legendre Gauss-Lobatto quadrature (SL-GLQ) points, a so-called shifted Legendre fractional pseudo-spectral differentiation matrix (FSL D-matrix) can be constructed (resp. integration matrix or FSL B-matrix) [28,29]. We will come back to the possible application of (shifted) Legendre polynomials in a next section (**¶ 3.2**.).

3. Application to Fractal Biological and Ecological Problems

3.1. The Mainardi Approach for Fractional Diffusion-Wave Equation

Two important hallmarks characterize the approach of Mainardi [2] for fractional diffusion and/or wave propagation: namely, the delineation of 'purpose' to provide the solution of the initial value problem (IVP), and the notion of 'causal function(s)'. Since the field variable $u = u(x, t; \beta)$ is assumed to be a causal function of time with $u(+\text{-}\infty, t; \beta) = 0$ for the following fractional partial differential equation:

$$\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = D\frac{\partial^2 u}{\partial x^2} \qquad \text{with } -\infty < x < +\infty \qquad \text{and } 0 < \beta \leq 1 \qquad (32)$$

where, for the sake of convenience, the order of time differentiation is denoted as 2β rather than α. The parameter D denotes a positive constant with the dimensions $L^2/T^{2\beta}$. Mainardi [2] distinguished mainly two cases, namely fractional diffusion, when $0 < \beta \leq 1/2$, and fractional wave-propagation, when $½ < \beta \leq 1$. In the former case, the equation must be equipped with a single IVP, say $u(x, 0^+; \beta) = f(x)$. In the latter case, two initial conditions are used, namely $u(x, 0^+; \beta) = f(x)$ and $(\partial u/\partial t =) u^{dot}(x, 0^+; \beta) = g(x)$.

Furthermore, $g(x) \equiv 0$, in order to ensure the continuous dependence of the solution on the parameter β, also in the transition from $\beta = (1/2)^-$ to $(1/2)^+$. Alternatives for the Dirichlet Boundary Condition will be discussed in the next section.

Mathematically, the assumption of causality implies that the output depends only on past and present, and not on future inputs. This allows for a causal function $\Phi_\lambda(t)$ to be defined as a recursive procedure:

$$\Phi_\lambda(t) := \frac{t_+^{\lambda-1}}{\Gamma(\lambda)} \qquad \text{with } \lambda \in \mathcal{C} \qquad (33)$$

and

$$t_+^{\lambda-1} = t^{\lambda-1}\Theta(t) \qquad (34)$$

and $\Gamma(\lambda)$ the gamma function (as described).

Moreover, the function $t^{\lambda-1}_+$ is locally absolutely integrable in the domain $0 \leq t < +\infty$, meaning that the Laplace transform of $\Phi_\lambda(t)$ can be calculated following [2]:

$$\mathcal{L}(\Phi_\lambda(t)) = \Phi_\lambda(s) := \int_0^{+\infty} e^{-st}\Phi_\lambda(t)\mathrm{d}t = \frac{1}{s^\lambda} \qquad \text{with Re } \lambda > 0 \qquad \text{and Re } s > 0$$

(35)

The convolution between two causal functions ϕ (t), ψ (t) is defined as:

$$\phi(t) * \psi(t) = \int_0^t \phi(t-\tau)\psi(\tau)\mathrm{d}\tau = \psi(t) * \phi(t) \tag{36}$$

whose Laplace transform is written as:
and

$$\mathcal{L}(\phi(t) * \psi(t)) = \bar{\phi}(s)\bar{\psi}(s) \tag{37}$$

$$\Phi_\lambda(t) * \Phi_\mu(t) = \Phi_{\lambda+\mu}(t) \qquad \text{with Re } \lambda > 0 \qquad \text{and Re } \mu > 0 \tag{38}$$

3.2. Use of (Shifted) Legendre Polynomials and possible alternatives for the Dirichlet Boundary Condition

The use of Legendre polynomials is derived from Legendre's differential equation (of order n) [30]:

$$(1 - z^2)\, Y'' - 2z\, Y' + n\,(n+1)\, Y = 0 \tag{39}$$

of which the general solution is:

$$Y = A\,\{1 - n(n+1)/(2!)\, z^2 + n\,(n-2)\,(n+1)\,(n+3)/(4!)\, z^4 - \ldots\} +$$
$$B\,\{z - (n-1)(n+2)/(3!)\, z^3 + (n-1)(n-3)(n+2)(n+4)/(5!)\, z^5 - \ldots\}. \tag{40}$$

If n is zero or a positive integer, polynomial solutions of degree n are obtained. These polynomial solutions are called the Legendre polynomials $P_n(z)$, n = 0, 1, 2, 3, …. By choosing these so that P_n (1) = 1, they were found to be expressed by Rodrigues' formula (named after Benjamin Olinde Rodrigues [1795-1851]):

$$P_n(z) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}(z^2 - 1)^n \tag{41}$$

from which: P_0 (z) = 1, P_1 (z) = z, P_2 (z) = ½ ($3z^2$ – 1), P_3 (z) = ½ ($5z^3$ – 3z), etcetera [30]. As described earlier (e.g. appendix b in [31]), these polynomial solutions are derived from Laplace's equation for diffusion on a sphere. The system defined by the expression P_n (cos θ) = 0 defines a set of nodal lines on the sphere, called harmonics, perpendicular to the axis and symmetrical to the diametral plane $\theta = \pi/2$ [31]. Moreover, for the physical interpretation of these harmonics, it is important to note that the value of any finite single-valued function of position on a spherical surface can be expressed as a series of rational integral harmonics, each of the form $r^n P_n$, provided the function has only a finite number of discontinuities and of maxima and minima on the surface [32]. From a realistic point of view, one may argue that ecological, global interpretations of these surface functions are only meaningful in absence of human interference, for instance in the seasonal migration patterns of birds. For the translation of these spherical harmonics to the realm of global migration and biodiversity fluctuations, whether or not influenced by anthropogenic activities [7], a closer look at the problem of Dirichlet Boundary Conditions and possible alternatives will be needed (see below).

An example of the use of Legendre-Galerkin approximation in so-called Sobolev spaces, e.g. for solving Burger's equation with Dirichlet Boundary Condition, was shown to be convergent in the interval [-1, 1] [33].

Also in the group of Abdelhakem [28,29], the Legendre polynomials are

presented as solutions for the Legendre differential equation, but by limiting the interval to [0,1], the polynomials are called 'shifted Legendre polynomials'(SLP), denoted by

$$P^*_m(z) = P_m(2z - 1) \tag{42}$$

Typically, the SLPs follow the well-known recurrence relations for Legendre polynomials [30], such as [29]:

$$(m+1)P^*_{m+1}(z) = (2m+1)(2z-1)P^*_m(z) - mP^*_{m-1}(z) \qquad \text{where } P^*_0 = 1, P^*_1(z) = 2z-1 \tag{43}$$

Moreover, SLPs are used as a trial function in the spectral analysis of the differential function $F(z)$ over [0,1] [28]:

$$\mathcal{F}(z) = \sum_{k=0}^{\infty} \Lambda_k P^*_k(z) \tag{44}$$

and, following the orthogonality requirements [29]:

$$\int_0^1 P^*_m(z)P^*_n(z) = \begin{cases} \frac{1}{2m+1} & \text{if } m = n, \\ 0 & \text{if } n \neq m. \end{cases} \tag{45}$$

Although the use of the Dirichlet Boundary Condition [9] is quite common, also other boundary conditions like the Neumann - (after Carl Neumann [1832-1925]), Cauchy - (after Augustin L. Cauchy [1789-1857]) and Robin - (after Victor G. Robin [1855-1897]) conditions have been proposed. The latter can be coined as combinations of the Dirichlet and Neumann Boundary Conditions. In the Neumann condition, not a specific value of the function is fixed at the boundary, but the value of the derivative at the domain boundary is defined (see e.g. Mainardi [2]). In the Robin boundary condition, a common expression for the insulating boundary condition for convection-diffusion equations is that the convective and diffusive fluxes at the boundary sum to zero.

According to Cantrell and Cosner [34], a Neumann Boundary Condition on a reaction-diffusion system (such as Fisher's equation, see above) can be interpreted as a 'reflective boundary', such that "all individuals encountering the boundary $\partial\Omega$ are reflected back onto Ω." In our approach of diffusion into - and across - a biodiversity interface (between several biotopes delineated by a natural fractal border) [15], such a hard-as-concrete border system is often not compatible with viable ecosystems (at both sides). And neither these interfaces can be regarded as 'insulating', such as in heat transfer modeling. Concrete boardings of channels instead of natural riparian border zones may cause frequent drowning of terrestrial mammals; or, concrete highway boardings form a lethal obstacle, for instance for the few remaining large feline carnivores in the Los Angeles region [35], and in many other cases. In agreement with common types of natural border zones, a Neumann-type Boundary Condition consisting of an exponential-logistic form is chosen (both at natural and anthropogenic discontinuities of the domain) [7]. For a partial differential equation, the Neumann boundary conditions on the domain $\Omega \subset \Re^n$ take the form:

$$\frac{\partial y}{\partial \mathrm{n}}(x) = f(x) = \nabla(y) \cdot \hat{\mathrm{u}}(x) \tag{46}$$

where **n** denotes the normal to the boundary $\partial\,\Omega$. The normal derivative ($\partial\, y/\, \partial$ **n** (x)) then is the inner product of the gradient vector of y (x) and the unit normal **û**. This gradient vector, moreover can be regarded as analogous to the dipole (bivector) notion

in a 5D-Clifford algebra (after William K. Clifford [1845-1879])(see ¶ **4. Heuristics for Resolving the 'Memory function' in Biodiversity Resilience**).

4. Heuristics for Resolving the 'Memory function' in Biodiversity Resilience

4.1. Memory in Ecosystems

The fractal geometry of nature [3] not only allows for a quasi-infinite miniaturization of the geological and landscape characteristics and life forms, it entails repetitiveness but also biodiversity and variability. In certain cases, the fractal geometry indirectly affects variation in time-averaged temperature profiles and physical exposure, such as in the rocky shorelines of the temperate climate zones (see ¶ **5. Support from Ecological observation data**). Differences in sensitivity in temperature and other physical characteristics, together with a stochastic principle of colonization, results in a primary variability and biodiversity at this natural border interface, which is especially prominent among the primary producers and life forms that depend on them [36].

For the 'higher' life forms, especially those that exert a top position in the food web, also other forms of niche-dependency are conspicuous. With increasing size and complexity of the organism, the aspect of roaming a larger area (for foraging) is obvious. It follows that the occurrence is the result of some kind of integration of the expected propensities of prey species in that area. Two types of 'memory' in ecosystems are discernable: a) a combination of deterministic and stochastic cues causing the occurrence of (mostly) lower life forms (including some form of hereditary determination); b) a causal effect of 'previous' acquaintance (either hereditary or acquired by some learning process) of the availability of food and shelter. The latter form of (causal) memory results in a pattern of 'stepping-stone' heuristics. The resilient occurrence of top predators for instance, requires the availability of one or more species of prey, as well as appropriate niches for sheltering, and in many cases also the absence of human disturbance and/or competitive predator species (see ¶ **5. Support from Ecological observation data**). On the other hand, the absence of certain communities or even of a single (dominant) species – the human species in particular - may cause a broken ecological chain, thus hampering the system's resilience.

4.2. Fibonacci and Stepping-Stone Heuristics

The stepping-stone heuristics form a quite popular concept that is related to the famous Fibonacci recurrence relation:

$$F_{n+1} = F_n + F_{n\text{-}1} \tag{47}$$

This Fibonacci recurrence relation was also invoked in the Petrov-Galerkin Lucas polynomials for the fractional-order derivative in the sense of Caputo [20], and indirectly in the use of shifted Legendre polynomials for fractional-order integration (and differentiation) [29,30]. Similarly, Podlubny [1] used a fractional-integral method for the description of 'hereditary' changes of mechanical properties in materials science, in particular in steel wires. Starting from a set of experimental measurements $y_1, y_2, \ldots, y_n$ a function $y(t)$ is fitted satisfying the integral equation:

$$y(t) = \sum_{k=0}^{m-1} a_k t^k - a_m \, {}_0D_t^{-\alpha} y(t) \qquad \text{with } 0 < \alpha < m \tag{48}$$

for which the constants α, a_k, ($k = 0, \ldots, m$) have to be determined, for instance

using the least square method. The parameters in the equation above allow physical interpretation: namely a_k (k = o,…, m-1) are initial values of the fitting function y (t) and its first (m-1) derivatives, whereby m is the smallest integer number exceeding α . "The fractional-order integral on the right-hand side represents the cumulative impact of the previous history of loading on the present state of the wire, and the order of integration (α), determines the shape of the memory function of the wire material" [1]. Then, by omitting the fractional-order integral one obtains the classical general polynomial regression model, and the problem is then reduced to the initial-value problem [1].

In the context of biodiversity dynamics, the Fibonacci numbers (F_i) would correspond to a biodiversity index with the nature of a probability density function (PDF), "based on the occupancy of ecological niches in an ecological phase space, identifiable with a set of dimensions of fractal nature" [6]. At the bottom of the ecological food web, a direct relation of organic growth with physical factors can be estimated and modelled, establishing a gradient-like propensity distribution of the primary species (see **¶ 5. Support from Ecological observation data**). The essential connection in interdependent ecological (food) chains, however, are the sheltering-foraging dipoles of both prey and predator species [6]. Biodiversity 'patterns' at a lower scale are thus remembered in higher order propensity distributions, as hunters have (acquired) knowledge of previous successful strategies and catches. The propensity distributions at a certain level of ecosystem organization, thus form a causal chain of interdependent biodiversity indices. These spatial distributions are mapped on a [0,1] probability range, indicating the complete absence or 100 % coverage of a certain biodiversity index [6].

4.3. Dipoles in Conformal Geometry

How can the ecological dipoles be expressed in a mathematical theory?

Formally, in 5D-conformal mapping theory, dipoles are defined as bivectors with a number of components dependent on the dipole's center, direction and radius (*). Under stereographic projection of the points thus defined, Euclidean space (resp. the x-y plane) is mapped to a parabolic volume (resp. paraboloid) [37]. In theory, the representation of a (global) biodiversity manifold could thus be represented by a

(*) Interestingly, in Conformal geometric algebra, a point in 3D- (or n-dimensional)-Euclidean space is constructed by adding two projective basis vectors (designated $\mathbf{e}_-$ and $\mathbf{e}_+$) to the set of ordinary basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots\}$, with the important properties that

$$\mathbf{e}_- \cdot \mathbf{e}_- = -1 \text{ and } \mathbf{e}+ \cdot \mathbf{e}_+ = +1.$$

From these basis vectors, two new vectors are conveniently defined, namely

$$\mathbf{e}_4 = ½ (\mathbf{e}_- - \mathbf{e}_+) \text{ and } \mathbf{e}_5 = \mathbf{e}_- + \mathbf{e}_+$$

A homogeneous point in 3D-Euclidean space then in conformal geometric algebra is expressed as:

$$p = x\,\mathbf{e}_1 + y\,\mathbf{e}_2 + z\,\mathbf{e}_3 + \mathbf{e}_4 + ½ (x^2 + y^2 + z^2)\,\mathbf{e}_5$$

When allowing the magnitude of (x, y, z) to become arbitrarily large, then $\mathbf{e}_5$ becomes identifiable with the point (direction) at infinity. It is important to note that also alternate definitions of $\mathbf{e}_4$ and $\mathbf{e}_5$ are possible,

in general: $\mathbf{e}_4 = a\,(\mathbf{e}_- - \mathbf{e}_+)$ and $\mathbf{e}_5 = b\,(\mathbf{e}_- + \mathbf{e}_+)$, a and b representing scalars. From these representations, in 5D-conformal geometric algebra, a bivector is defined (having a center at p, a line direction and a radius or norm, resulting in ten components altogether).

Theoretically, the stereographic projection procedure applied to the biodiversity manifold, may represent interdependent dipoles with varying direction and radius (following conformal geometric-algebraic rules) [37].

5D-conformal mapping, in accordance to Riemann's mapping theorem [9,30,37].

A physical correlate of conformal mapping has been suggested to manifest itself in the occurrence of fractal patterns in biological and geological systems, that are also seen in so-called Mandelbrot figures (which correspond to the Mandelbrot set of points in the complex plane) [6]. In particular, the fractional transformation $w = (\alpha z + \beta)/(\gamma z + \delta)$, with $\alpha\delta - \beta\gamma \neq 0$, can be considered as combinations of the transformations of translation, rotation, stretching and inversion [30]. Various examples of conformal mappings according to Riemann's theorem (e.g. from the half plane on to a circle disc, or from the exterior/interior of a parabola on to a circle in the complex plane) are given in [30]. Similarly, in the case of ecological dipoles, where the half plane is an abstract representation of space-time (with only one spatial dimension), the same approach may be followed, which so far is lacking in ecological literature (to our present knowledge). In order to expand this conformal mapping approach to the complex multidimensional problem of ecosystem development, obviously, further research is needed, which surpasses the scope of the present paper. However, reduction of the complex ecological case to fewer dimensions may help in clarifying the picture.

Interestingly, the PDF (in one spatial dimension) has been defined in so-called continuous time random walk models (CTRW) [13], following the use of the Fourier-Laplace transforms and the convolution theorem for the integral equation:

$$P(x,t) = \int_0^1 \eta(x,t')\,\omega(t-t')\mathrm{d}t' \tag{49}$$

$$\text{where} \qquad \omega(t) = 1 - \int_0^t \phi(\tau)\mathrm{d}\tau \tag{50}$$

is "the probability of not having made a jump until time t, and η (x', t') is the PDF of just having arrived a position x' at time t' < 1" [13]. According to these authors, the result of the Fourier-Laplace transform, and using the Dirac delta function (δ) for calculating the jump length following a time t, is a useful algebraic relation for implementing into an iterative procedure (after [38]).

In order to infer a global fractal biodiversity approach from local PDF descriptions, however, a large number of causative mechanisms, whether or not influenced by anthropogenic disturbances, have to be unraveled and corroborated by ecological field observations [7]. Also, certain mathematical characteristics of specific polynomial functions, such as the orthogonality of (shifted) Legendre (and other) polynomials, are not known how they are fulfilled in their ecological counterparts. Or, the well-known expression for the shifted Legendre polynomials [29]:

$$\int_0^1 P_m^*(z)P_n^*(z)\mathrm{d}z = 0 \qquad \text{for } n \neq m \tag{51}$$

P^*_m and P^*_n here representing successive network stadia, is only fulfilled provided they are representing the product of their vector projections on an orthogonal basis. So, the biodiversity PDF descriptions must be projected orthogonally. The orientation of the foraging-sheltering dipoles, and more in general, the dominant direction of e.g. bird migration routes (see ¶ **5.2**.), could be regarded as an example of spatially directed evolvements. Or, a more abstract parametrization, such as in principal component analysis, may be preferred for describing the successive stadia. Hereby, a parallel may be drawn to the diffusing diffusivity approach [23] when applied to fluctuations in biodiversity (see ¶ **2.1**).

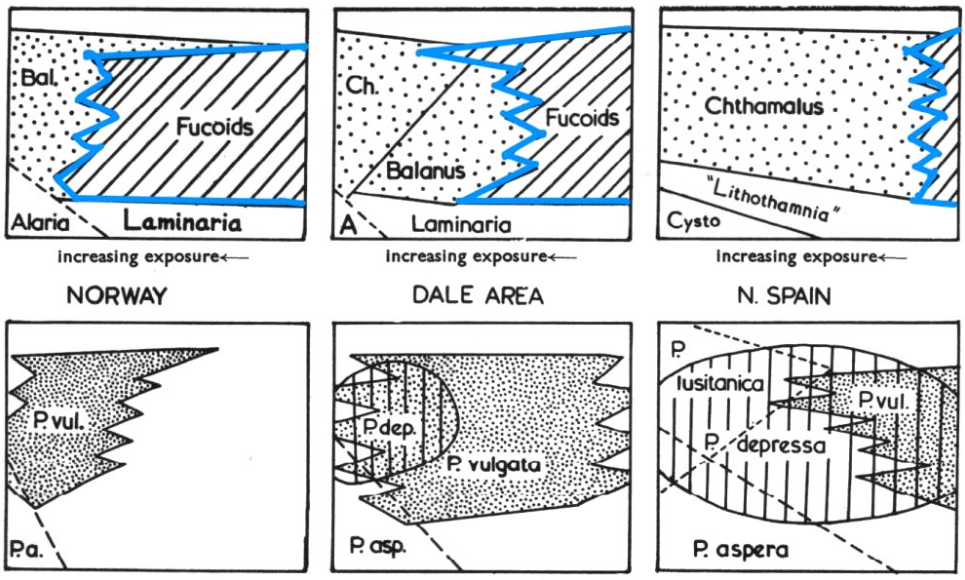
To conclude this paragraph, in the complex conditions of ecosystem network modelling, the use of Legendre polynomials, as well as of other methods of orthogonal polynomials, will require further fundamental research and observational data acquisition too (see also next section).

5. Support from Ecological observation data

In order to support the fractal biological and geophysical conditions that have an important repercussion on the mathematical models discussed in the previous paragraphs, three levels of biological organization are discerned: **a)** the primary producers/consumers and pioneering species; **b)** the seasonal flocks of bird species and other migratory species, and **c)** the top-predators. An abstract parametrization of these three levels is depicted in Fig.3 (see **¶ 6. Concluding remarks and summary**).

5.1. Primary producers/consumers and pioneering species

The littoral gradients depicted in **Fig. 1** are obtained from field observations made at rocky sea shores in Europe, from Norway to Northern Spain [39]. Each diagram represents a gradient (based on a series of examined shores in the area) from 'exposed' to the left and 'sheltered' at the right. The top of each diagram represents the top of the midlittoral zone and the bottom passes into the infralittoral fringe [39].



Legend to Fig. 1. – Changes in so-called 'Barnacle – Fucoid Balance' (*blue line*) (*) from north to south locations (*top row*), associated with changes in distribution of *Patella spp*. (Common Limpets; Gastropoda) (*bottom row*) at the same locations. (Adapted from: Ballantine, 1961) [39].

(*) *Top row*: Barnacles are sedentary Crustaceans, e.g. *Balanus, Chthamalus*. On the other hand, *Alaria, Laminaria* and also *Cystoseira* [in so-called *Cysto* zone] are all brown seaweeds (Phaeophyta, Algae), whereas *Lithothamnia* is a red seaweed (Rhodophyta, Algae) species. Dale is in the Pembrokeshire area in Wales, UK. (See [39] and main text.)

The diagrams show the dependence of the littoral species on the degree of exposure (to sea, water tides and wave actions, following the sea action and the geological patterns of rocky shores). The areal pattern appears to be defined by the average ambient temperature. Moreover, a stochastic seeding pattern is prominent for the barnacles, sea weeds and sea snails. Barnacles and Common Limpets are animal

species (Crustacea, resp. Gastropoda) that graze fucoid sporelings and that "thrives best on the edge of the fucoid : barnacle boundary" [39]. For, Fucoids (brown sea weeds) "tend to smother barnacles and barnacles compete with limpets for space and reduce their feeding efficiency" [36].

As shown in **Fig. 1**, all species examined behave in the same way occupying increasingly sheltered shores towards the southern end of their (geographical distribution) range, which in this example is Northern Spain [36,39]. However, alternative explanations exist, for instance that "a greater proportion of sheltered shores in the north may produce a recruitment advantage to sheltered shore organisms and push the exposure-loving species further towards exposure or even eliminate them altogether" [39]. This may indirectly result in an increased biodiversity, for, extreme ambient conditions tend to favorize the survival of species that are specific to these extremes.

A direct power law dependence of growth rate and ambient temperature for instance has been measured in laboratory conditions for the barnacle (*Balanus balanoides*) [40]. Similar laboratory results were also obtained for estimating the growth rate of some dominant coral reef species [41].

Mathematically, a given distribution in theory can be represented by a propensity matrix of N species and M geophysical and coordination parameters. An important difference of the probability density function with such a propensity matrix [42] is that no longer molecular species and chemical reactants are represented, but random seedings, local growth and consumption rates. The dipole notion moreover is useful, in order to denote the growth direction (at a local scale) starting from a 'moving border zone', till the foraging-sheltering dipoles of migrating birds (see 5.2.).

5.2. Birds and other seasonally migrating species

Literally dozens of migratory bird species are specialized in catching food from the littoral fauna and flora, albeit the rocky shores (e.g. Oystercatchers, *Haematopodidae*) or sandy and muddy shores (like Curlew, Red Knot and many other *Scolopacidae, Charadriidae*). Their species-specific migratory patterns not only follow the seasonally available foraging grounds, but also follow the presence of suitable terrains for nesting, breeding and sheltering their offspring [43,44]. The geographical range of their migration routes may spread from northern Russia to the North Sea [43], or even span routes from southern to northern hemispheres (e.g. [44]). Obviously, also many other animal groups show a seasonal, cyclic migration pattern, e.g. whales, sea turtles, et cetera. The cyclic pattern of sea turtles searching for their breeding shores, moreover, causes resident populations of top-predators like the Jaguar (*Felis onca*), for instance in Costa Rica [45], to temporarily hunt the sea turtles and to move their foraging territory towards the shores. In turn, these migrations cause other species in the food web to migrate too, such as the Coyote (*Canis latrans*), a so-called meso-predator, because of the seasonal absence of the top-predator [45].

Seasonal migration patterns can be represented as a geographical pattern on the globe by fitting a finite number of harmonic functions, that can be formulated as a multiple of infinite exponential series [46]. The dynamic changes in distribution pattern of one species, especially these at the top of the food chain (see also 5.3), directly affect the dynamic patterns at intermediate levels, and further on downwards the food chain.

5.3. Top-predators

For the recent, apparent logistic/exponential growth (see **Fig. 2**) of some

populations of top-predators, a direct or hidden influence of anthropogenic activities is often present. This may result from the restrained presence of prey species in human settlements and reserves or from the actual feeding behavior by direct human interference [7]. The exceptionally sharp rise in the numbers of breeding pairs of White-tailed Eagles (*Haliaeëtus albicilla*)(**Fig. 2**, below), at the beginning of the millennium, reportedly resulted from the high prevalence of carcasses of Red Deer (*Cervus elaphus*), that died in some fenced territories (because of their high reproduction and the shortage of proper food) [47]. On the other hand, the marked rise of the Golden Eagle in Switzerland (as well as in other parts of Europe) in the twentieth century, resulted from protection measures. Whereas the hunt of these top-predators has been prohibited, the use of leaded munition for the hunt of some prey species, has resulted in several cases of lead poisoning and a decline (up to 30 %) of their reproductive success [48].

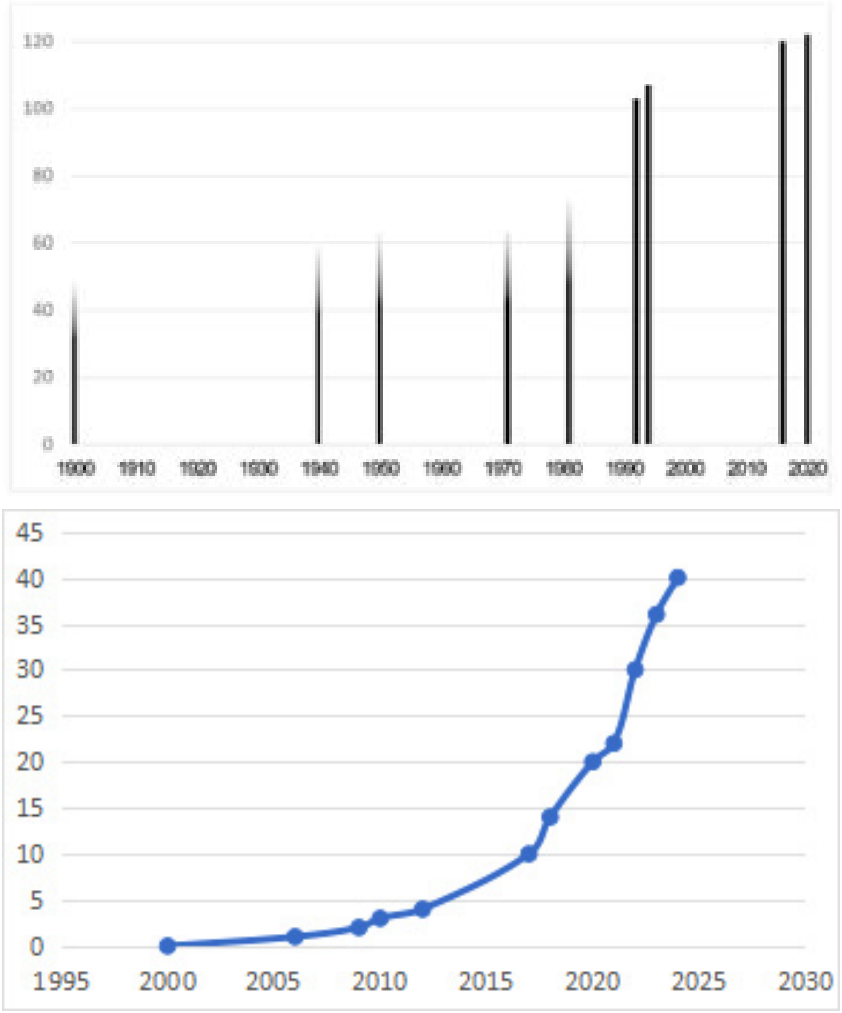


Fig. 2. Growth of the number of breeding pairs of some top-predators: the Golden Eagle (*Aquila chrysaëtos*) in Switzerland (*top*) and White-tailed Eagle (*Haliaeëtus albicilla*) in the Netherlands (*below*). Whereas the Golden Eagle has been present in the Alps since historical recordings, the breeding of White-tailed Eagle in the Netherlands is a relatively new phenomenon (see also main text) (data from [48], resp. [49]).

Current opinion tells that the presence of top-predators is beneficial to and normally increases the biodiversity of the supporting prey populations. On the other

hand, unlimited growth of (especially fast reproducing) top herbivores in absence of sustainable predator populations cause ecosystems to collapse [47]. Important questions remain though, for instance what happens with supporting populations of prey communities, when a forced increase of top-predators is imposed by anthropogenic interference.

6. Concluding Remarks and Summary

To conclude this paper, we have summarized an abstract representation of a typical set of ecosystem data in **Fig. 3**. The ecosystem represented is derived from the rocky shore example depicted in **Fig. 1**. Stereographically, three zones are represented starting from the border interface, in this case a geophysical tidal border zone (left), an intermediate zone and an evolved so-called succession zone (right). A similar representation is conceivable for coral reef and mangrove border interfaces, as well as any other natural border system.

The physical processes corresponding to these zones partially overlap, and can be summarized as: 1) diffusion, growth and chemical attraction (from input of sea nutrients and floating eggs, larvae, sea plankton, etc.) (red arrow indicates growth direction); 2) growth, dispersion and competition between primary producers and consumers (represented by green, blue and yellow symbols); 3) diffusion-wave and roaming zone of predators. Two series of successive communities (a collection of species) are represented by the P_1, P_2, P_3,… and Q_1, Q_2, Q_3,… symbols. These successions generate distinct ecosystem patterns in time and space.

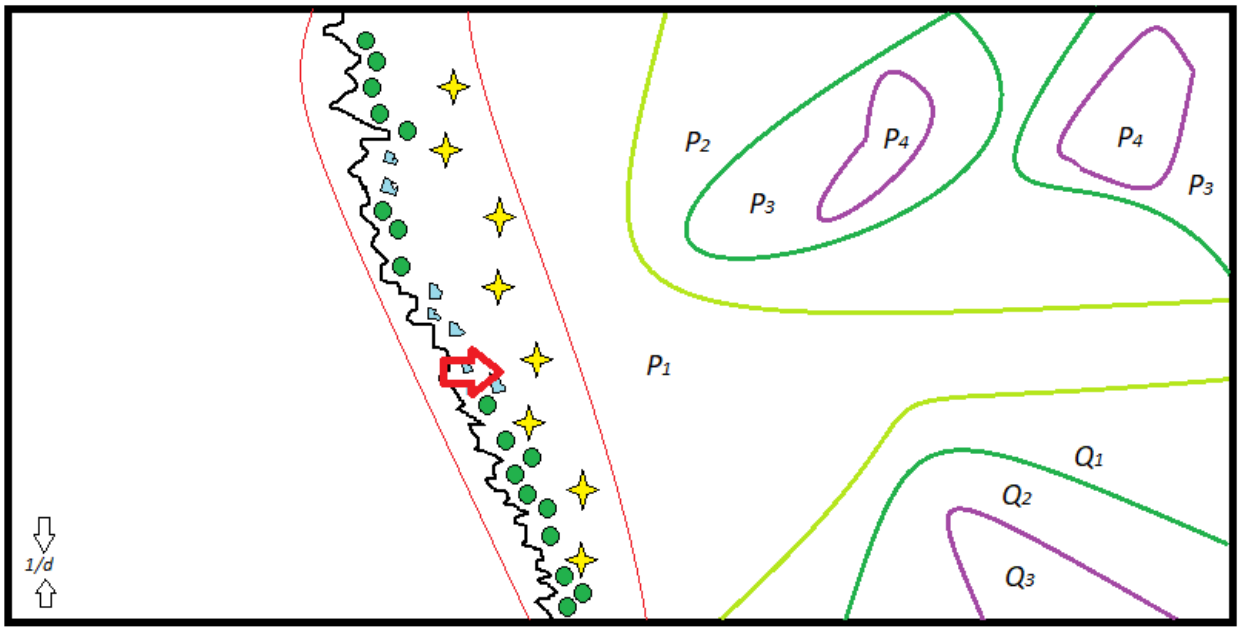


Fig. 3. Abstract parametrization of the tidal-littoral ecosystem (see also **Fig. 1**), subdivided in a geophysical (tidal) border (left), an intermediate and succession zone (right). See main text for explanation of symbols. ($1/d$ is the fractal dimension, based on the fractal appearance of the border zone, represented at a relevant scale for the developing organisms).

The mathematical formulas corresponding to these combined physical processes are, in slightly simplified form, for the border zone:

$$\frac{\partial P(n_1, n_2, n_3, \cdots a_1, a_2, a_3)}{\partial t} = D_{ni}\frac{\partial^2 n_i}{\partial x^2} - (\nabla n_i \cdot \nabla a_i) + D_{ai}\frac{\partial^2 a_i}{\partial a_i^2} \tag{52}$$

with n_1, n_2, n_3,.. the abundancies of (dominant) species, and a_1, a_2, a_3, … are concentrations of key attractant factors.

For the intermediate and succession zones :

$$\frac{\partial P_{top}}{\partial t} =_0 D_t^{\alpha} \sum_i \int_0^t (P_m + P_n + P_o) s_{i-1}(\tau) s_{i-1}(t-\tau) \mathrm{d}\tau \tag{53}$$

with P_m, P_n, P_o, … the probability density functions of different (but concomitant) species communities, and P_{top} that of a top predator (in the absence of human interference) [7].

Mathematically, most of these processes are described in literature, especially in the classical diffusion-chemotaxis studies (see ¶ **2.1**.), but also fractional calculus has been introduced for describing aspects of the ecosystem dynamics. In this paper, we have focused on the application of fractional calculus, for instance in the Mainardi approach of diffusion-wave processes (see ¶ **3.1**.). Important questions on the problem of boundary conditions remain to be solved for specific cases, following the methods of Dirichlet, Neumann, Robin and possible combinations (see ¶ **3.2**.).

An important finding is that in ecosystem progression two types of causal relations occur. Therefore, also a combination of iterative procedures is discerned, namely the purely stochastic-deterministic mechanism, and a procedure following some hereditary heuristic, whereby the memory of past combinations of encounters is directing the evolvement of the 'top' of the ecosystem. Relatively new methods introduced here are the use of shifted Legendre polynomials in spectral analysis, and the bivector notion inferred from conformal geometric calculus.

Another important finding is that applying fractal descriptions to natural ecosystems, implies that the fractal appearance at a miniaturized scale not only causes miniaturization and self-repetition [6], but also allows for diversification due to the occurrence of physically divergent micro-climates and stochastic distributions of distinct species of organisms (with divergent micro-climate preferences)(see ¶ **5**). This is especially the case in the rocky tidal zone (**Fig. 1**), in coral reefs and in similar border interfaces. Finally, this implicates that the fractal geophysical conditions of the border interface generate the miniaturization and biodiversity of the entire ecosystem, starting from the bottom of the ecological web, but with possibly divergent outcomes for the so-called top layers.

Conflict of interest: The author declares no conflict of interest.

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